V-OFDM: On Performance Limits over Multi-path Rayleigh Fading Channels

Peng Cheng, Meixia Tao, Yue Xiao, and Wenjun Zhang

Abstract

As a bridge of connecting orthogonal frequency division multiplexing (OFDM) with single-carrier frequency domain equalization (SC-FDE) techniques, Vector OFDM (V-OFDM) provides significant flexibility in system design. This paper presents an analytical study of V-OFDM over multi-path fading channels. Our goal is to investigate the diversity gain and coding gain of each vector block (VB) in V-OFDM so as to ultimately reveal its performance limits over fading channel. By using algebraic number theory tools, we prove for the first time that a majority of VBs in V-OFDM can surely realize the diversity gain of \( \min\{M, G\} \), where \( M \) is the length of each VB, and \( G \) is the total number of channel taps. Furthermore, some specific VBs, whose length equals the total number of channel taps, can not only harvest the maximum diversity gain but also achieve the maximum coding gain. It is further demonstrated that, even though VBs fail to benefit from additional diversity gain when \( M \) exceeds \( G \), they can enjoy significantly increased coding gains. Our analysis concludes that it is preferable to choose the length of VBs to be equal to the number of channel taps in consideration of both overall system performance and computational complexity.

Index Terms

OFDM, V-OFDM, pairwise error probability, diversity gain, coding gain, algebraic number theory, cyclotomic fields
ORTHOGONAL frequency division multiplexing (OFDM) [1]-[3] is a promising multicarrier transmission technique for high-speed wireless communications. It has been adopted for a number of standards such as HIPERLAN/2, IEEE802.11a, and is also a key technology of IEEE802.16e (WiMAX) and 3GPP Long Term Evolution (LTE) and LTE-Advanced downlink systems [4], [5]. The significance of OFDM system when operating in wireless channels is to be able to transform a frequency-selective fading channel into parallel flat-fading subchannels, thereby effectively reducing the receiver complexity. However, this comes at a cost for the loss of multi-path diversity. This is due to that each symbol is only transmitted over a single flat subchannel; reliable detection of those symbols carried by the subcarriers in null location of the frequency band becomes difficult, if no channel coding is applied. Another main limitations associated with OFDM is the high peak-to-average power ratio (PAPR) of the transmitted signals [6] and high sensitivity to carrier frequency offset (CFO) [7]. Single-carrier frequency domain equalization (SC-FDE) [8], [9] is an alternative approach to OFDM with reduced PAPR and CFO. As an extension of SC-FDE, single-carrier frequency division multiple access (SC-FDMA) technique is employed in the uplink of LTE framework as a substitution for OFDMA to improve the power efficiency and reduce manufacturing cost [10]. However, SC-FDE also suffers from a drawback that the transmitter and receiver have unbalanced complexity. Moreover, when incorporating multiple-input multiple-output (MIMO) technique, the frequency-domain equalization would be much more complex than OFDM. In addition, multi-user diversity can be easily obtained in OFDM via distributed subcarriers allocation. As such, SC-FDE is not very suitable to be applied in downlink for high speed transmission in contrast to merits of OFDM.

Vector OFDM (V-OFDM), as a generalization of OFDM, first proposed by Xia in [11], [12] to combat frequency null, serves as an important bridge between OFDM and SC-FDE. In contrast to OFDM, V-OFDM is able to convert an intersymbol interference (ISI) channel into an ISI-free vector channel while involving channel matrices instead of channel coefficients in one-tap equalization so as to increase the diversity order. Further, V-OFDM has a smaller PAPR and is less sensitive to CFO than OFDM. Because of the shift in operation from receiver to transmitter, the computational complexity and thereby, the power consumption of SC-FDE at the receiver will be relieved in the case of large bandwidth configuration.

Following the advent of V-OFDM, V-OFDM with adaptive vector channel allocation are introduced in [13]. Afterwards, synchronization and guard-band-configuration for V-OFDM are demonstrated in
Further studies about iterative decoding and demodulation in V-OFDM through turbo principle are presented in [15]. The recently proposed asymmetric OFDM [16] is actually in parallel with V-OFDM, but is based on different mathematical derivation. Almost all of these research advances find and conclude that, under frequency-selective fading channels, the bit error rate (BER) performance can be notably improved by increasing the dimension of vector block (VB) as a benefit from signal space diversity [18]. In contrast to time, frequency and spatial diversity, which requires redundancy and thereby consuming more resources in power and/or bandwidth, signal space diversity serves as a power and bandwidth efficient mean to resist multi-path fading. However, the above result and conclusion have not been analytically proved explicitly yet. In specific, references [14] and [15] only present some qualitative recommendation and simulation results. Nevertheless, they cannot be regarded as a sufficient and persuasive revelation if without rigorous analytical study. For instance, one may naturally doubt whether increasing VB length would always improve the performance without any limitations. Whether V-OFDM can truly benefit from signal space diversity gain and what is the inherited limitation associated with V-OFDM remain unknown. More recently, in [19], the authors proposed a method based on constellation rotation to improve the BER performance of V-OFDM. However, it is found that with this method the individual diversity properties for each VB in V-OFDM are lost. As such, the specific properties of original V-OFDM system is still obscure.

In this paper, a comprehensive analytical study of V-OFDM under fading channel is presented. Our goal is to investigate diversity gain and coding gain for each VB. We first deduce the symmetrical characteristic of VBs. By employing algebraic theoretic tools, we certify for the first time that almost all the VBs (need to exclude some VBs) of V-OFDM are able to achieve the diversity gain of $M$, where $M$ is the length of each VB, but this diversity gain is strictly limited by the total number of channel taps. As a result, it is reasonable to expect that the performance of V-OFDM should be considerably improved in terms of increased $M$ with sufficient channel taps. We further show that among those VBs achieving diversity order of $M$, only several VBs can achieve the maximum coding gain and some may have extremely small coding gain. Therefore, due to the fluctuation of the coding gains for each VB, V-OFDM will inevitably suffer from some performance loss. Furthermore, when $M$ exceeds the number of channel taps, further increasing $M$ cannot contribute to the increase of the diversity gain; however, we find that the coding gain here can be significantly promoted so that quite a few of VBs can achieve the maximum coding gain. As a result, the performance of V-OFDM is very close to its limit. We conclude that choosing the
length of VBs to be equal to the number of channel taps strikes a good balance between overall system performance and computational complexity at the receiver. Our analytical results indicate that V-OFDM can definitely benefit from signal space diversity gain over multi-path fading channel.

The remainder of this paper is organized as follows. In Section II, the vector OFDM system model is reviewed. In Section III, we derive the performance limits of V-OFDM in terms of the diversity gain and coding gain. In Section IV, we discussed similarities and differences between V-OFDM and some other system models. Illustrative analysis and simulation results are presented and discussed in Section V. Finally, we conclude this paper in Section VI.

The notations used in this paper are as follows: vector and matrix are denoted by symbols in boldface, and $(\cdot)^T, (\cdot)^H, (\cdot)^*, (\cdot)^{-1}, \text{Tr}(\cdot)$, and $\text{rank}(\cdot)$ denote the transpose, the conjugate transpose, the conjugate, the inverse, the trace, and the rank operation, respectively. $\text{det}(\cdot)$ denotes the determinant of a square matrix. $I_M$ is the $M \times M$ identity matrix and $\text{diag}(a_1, \cdots, a_M)$ is the diagonal matrix with element $a_m$ on the $m$-th diagonal. $\|\cdot\|_2$ represents the 2-norm of a vector. $CN(0, \sigma^2)$ denotes the complex normal distribution with independent real and imaginary parts each with mean zero and variance $\sigma^2/2$. $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ stand for the positive integer set, the integer ring, the rational number field, the real number field, and the complex number field, respectively. $\mathbb{C}^{M \times N}$ represents the set of $M \times N$ matrices in complex field. $\mathbb{Z}[j]$ denotes the algebraic integer ring, with elements $p + jq$ where $p, q \in \mathbb{Z}$, and $\mathbb{Q}(j)$ is the smaller subfield of the set of complex number $\mathbb{C}$, including both $\mathbb{Q}$ and $j$. Euler’s totient function $\phi(P)$ is defined as the number of positive integers that are less than $P$, and relatively prime to $P$. $\gcd(a, b)$ represents the greatest common divisor between $a$ and $b$. All the subscript indications in this paper begin with zero. Besides, to facilitate the reading, we list the major variables used throughout this paper in Table I.

II. SYSTEM DESCRIPTION

Firstly, for a conventional OFDM system, let $\{S_0, S_1, \cdots, S_{N-1}\}$ ($S_i \in \Phi$ and $\Phi$ stands for a conventional two-dimension constellation of size $2^\varphi$), $\{R_0, R_1, \cdots, R_{N-1}\}$ and $\{W_0, W_1, \cdots, W_{N-1}\}$ denote $N$ transmitted subcarriers, $N$ received subcarriers and additive white Gauss noise, separately. $H = \{H_0, H_1, \cdots, H_{N-1}\}$ characterizes the corresponding frequency response on $N$ subcarriers. Then it is well-known that the following input output relationship can be established with proper attached cyclic prefix (CP) [12]:

$$R_k = H_k S_k + W_k, \ k = 0, 1, \cdots, N - 1.$$  

(1)
A. Transmitter Model

V-OFDM can be regarded as a general vectorization of OFDM, for which the concept of subcarrier is substituted for vector block (VB). The transmitter diagram is shown in the upper half of Fig. 1 (a simple example is also embedded in this figure), where a serial of modulated signals \( \{S_0, S_1, \cdots, S_{N-1}\} \) are first column-wise blocked into an \( M \times L \) matrix \( S \), where \( N = M \times L, N = 2^{n_0}, M = 2^{m_0}, L = 2^{l_0}, n_0, m_0, l_0 \in \mathbb{N} \) and its entries are \( [S]_{m,l} = S_{lM+m}, m = 0, 1, \cdots, M - 1, l = 0, 1, \cdots, L - 1 \). At this point, each column \( S_l \in \mathbb{C}^{M \times 1} \) of \( S = [S_0, S_1, \cdots, S_{L-1}] \) is referred to as an original frequency VB at the transmitter. Then, the inverse fast Fourier transform (IFFT) of size \( L \) is performed over each row of \( S \in \mathbb{C}^{M \times L} \) to yield a matrix \( s \in \mathbb{C}^{M \times L} \) in the time domain as

\[
s = SF_L^{-1}
\]

where \( F_L^{-1} \) denotes an \( L \)-point normalized IFFT matrix whose \((k, n)\)-th entry is defined as \( [F_L^{-1}]_{k,n} = L^{-1/2} \exp \left( j2\pi kn/L \right) \). Afterwards, each column \( s_l \in \mathbb{C}^{M \times 1} \) of \( s = [s_0, s_1, \cdots, s_{L-1}] \) is referred to transmitted vector, corresponding to original VB \( S_l \) in the frequency domain. Before transmitting, the last \( P \) vectors of \( \{s_0, s_1, \cdots, s_{L-1}\} \) (\( PM \) symbols) are inserted as CP vectors, namely,

\[
\{s_{L-P}^T, s_{L-P+1}^T, \cdots, s_{L-1}^T, s_0^T, s_1^T, \cdots, s_{L-1}^T\}
\]

are transmitted serially through the channel in the symbol order.

B. Receiver Model

The frequency-selective channel is modeled as a \((G - 1)\)-th order discrete-time baseband impulse response vector, denoted as \( h = [h_0, h_1, \cdots, h_{G-1}]^T \), where \( h_l \) stands for the \( l \)-th channel tap and each \( h_l \) is modeled as independent and identically distributed (i.i.d) random variables according to \( \mathcal{CN}(0, 1/G) \). Here, zero-padding \( h \) is denoted as \( \overline{h} = \left[ h_0, h_1, \cdots, h_{G-1}, h_G, \cdots, h_{N-1} \right]^T \) and \( \overline{h}^T \) is column-wise blocked into a \( M \times L \) matrix \( H \) with \( [H]_{m,l} = h_{lM+m}, m = 0, 1, \cdots, M - 1, l = 0, 1, \cdots, L - 1 \) for notational conventions in the following.

At the receiver, let \( \{r_0, r_1, \cdots, r_{N-1}\} \) denote the \( N \) received symbols after removing CP. An inverse operation as in the transmitter is performed to extract the original data symbols. The receiver diagram is depicted in the lower half of Fig. 1 and is outlined as the following steps:
1) Column-wise block \( \{r_0, r_1, \cdots, r_{N-1}\} \) into a \( M \times L \) matrix \( r \) and its entries are \( [r]_{m,l} = r_{lm+m}, m = 0, 1, \cdots, M - 1, \ l = 0, 1, \cdots, L - 1. \)

2) Take FFT operation over every row of \( r \in \mathbb{C}^{M \times L} \) to obtain a matrix \( R \in \mathbb{C}^{M \times L} \) in frequency domain as \( R = rF_L. \)

3) Denote \( R = [R_0, R_1, \cdots R_{L-1}] \) and each column \( R_l \) of \( R \) represents a received VB corresponding to the original VB \( S_l \) at the transmitter.

C. Characteristics of signal input and output

As noted in [12], [14], [15], and [16], when CP length is any integer larger than the order of the channel taps, namely, the condition \( PM \geq G - 1 \) is satisfied, the vector OFDM system provides ISI-free vector channels in the sense that

\[
R_l = H_l S_l + W_l, \ l = 0, 1, \cdots L - 1
\]

in which \( W_l = [w^T_l 0, w^T_l 1, \cdots, w^T_{l M-1}]^T \) denotes the noise vector whose entries are i.i.d. random variables according to \( \mathcal{C}\mathcal{N}(0, N_0) \). Each \( H_l \in \mathbb{C}^{M\times M} \) characterizes a vector channel; detailedly, \( H_l \) takes the specific form as (see [12, Eq. 3.6, Eq. 3.11], [14, Eq. 7], [16, Eq. 10] and [19, Eq. 5])

\[
H_l = \begin{bmatrix}
H_{0,l} & \exp\left(-j\frac{2\pi m}{L}\right) H_{M-1,l} & \cdots & \exp\left(-j\frac{2\pi}{L}\right) H_{1,l} \\
H_{1,l} & H_{0,l} & \cdots & \exp\left(-j\frac{2\pi}{L}\right) H_{2,l} \\
\vdots & \vdots & \ddots & \vdots \\
H_{M-1,l} & H_{M-2,l} & \cdots & H_{0,l}
\end{bmatrix}
\]

(4)

in which \( [HF_L]_{m,l} = H_{m,l} \). Furthermore, we can find that \( H_l \) is a factor-circulant matrix [20] and an important property is that it can be diagonalized as [14, Eq. 22]

\[
H_l = U_l^H \hat{H}_l U_l
\]

(5)

in which \( \hat{H}_l \) will be explained latter and \( U_l \in \mathbb{C}^{M \times M} \) is a unitary matrix whose entries \( [U_l]_{s,m} \) can be illustrated as

\[
[U_l]_{s,m} = \frac{1}{\sqrt{M}} \times \exp\left(-j\frac{2\pi}{N}(l + sL)m\right), \ s, m \in \{0, 1, \cdots, M - 1\}.
\]

(6)
It is straightforward to show that $U_l$ in (5) can be written as a more compact form

$$U_l = F_M \Lambda_l, \quad \Lambda_l = \text{diag} \left( 1, \gamma_l, \cdots \gamma_l^{M-1} \right)$$

(7)

with $\gamma_l = \exp \left( -j2\pi l/N \right)$. On the other hand, $\hat{H}_l$ (c.f. (5)) is directly correlated with frequency response, denoted as $\{H_k\}_{k=0}^{N-1}$ in conventional OFDM. Recall that in conventional OFDM (c.f. (1)), frequency responses $\{H_k\}_{k=0}^{N-1}$ are the zero-padded $N$-point FFT (without normalization) of the channel impulse response vector $h$ defined as before. Then for V-OFDM, $\hat{H}_l$ in (5) is a diagonal matrix whose diagonal elements consist of channel sampled frequency response vector obtained from $\{H_k\}_{k=0}^{N-1}$ in conventional OFDM. That is

$$\hat{H}_l = \text{diag} \left( H_l, H_l+L, \cdots, H_l+(M-1)L \right).$$

(8)

Overall, the structure of V-OFDM is simple, only requiring $M L$-point IFFT/FFT operations combined with some VBs partition. The major differences between V-OFDM and conventional OFDM stem from the fact that V-OFDM involves channel matrices instead of channel coefficients in one-tap equalization as in conventional OFDM system. Since $H_l$ can be diagonalized as in (5), $U_l$ herein can be viewed as a rotated matrix in view of the equivalent mathematical form, regardless of more specific physical significance. V-OFDM can be generally thought of as a combination of rotated signal space with rotated matrix $U_l$, as in [18]. Different from time, frequency and space diversity, which requires redundancy and thus consumes extra resources in power and bandwidth, signal space diversity is power- and bandwidth-efficient [18]. The main idea in signal space diversity is to treat $K$ transmitted symbols jointly as a constellation carved from an $K$-dimensional lattice, and then a rotation matrix is applied to the lattice constellation in order to optimize the diversity order. However, whether V-OFDM can truly benefit from signal space diversity under multi-path channel, to date, is unknown.

In [21] and [23], the authors use the metrics of diversity gain and coding gain to measure the performance of precoding based OFDM and space-time diversity systems. By regarding each VB as a different precoding matrix $U_l$, this motives us to investigate the diversity gain and the coding gain for each VB so as to effectively weight the overall performance of V-OFDM. It should be mentioned that, we cannot directly equate V-OFDM with precoding based OFDM [21], [22] in that the transmitted subcarriers of V-OFDM are only partially orthogonal. We will give a detailed comparison between V-OFDM and precoded OFDM in the Section V.
By observing (3) and (5) closely, it can be realized that OFDM and SC-FDE can be regarded as special cases of V-OFDM with $M = 1$ and $M = N$, respectively. To be specific,

- **OFDM.** By setting $M = 1$ and $L = N$, in this case $U_l = 1$ and (3) becomes $R_l = H_l S_l + W_l$, $l = 0, 1, \cdots, N - 1$, which indicates the $l$-th OFDM subcarrier output.

- **SC-FDE.** Let $M = N$ and $L = 1$, which means that there is only one VB. In this case $U = F_M$ (c.f. (7)) and (3) becomes $R = F_M^H \hat{H} F_M S + W$, where $F_M^H \hat{H} F_M$ equals a circulant matrix $H$.

Resultingly, it shows a CP based single-carrier block transmission system. If a followed FFT operation is connected, $R' = \hat{H} F_M S + W'$ expresses a SC-FDE.

Therefore, V-OFDM can be vividly regarded as a bridge connecting OFDM and SC-FDE, which provides significant flexibility in system design. Despite that OFDM and SC-FDE have been extensively analyzed in the literature, the properties of the bridge, V-OFDM, are much less investigated; thus, it is worthwhile to further exploit and analyze the characteristics of V-OFDM.

### III. Performance Analysis

#### A. Pairwise Error Probability

BER is a common performance measure in wireless systems. But its explicit and closed-form expression may not be tractable. In many occasions, pairwise error probability (PEP) affords a good approximation for BER performance at high signal-to-noise ratio (SNR) and therefore has been applied extensively. In this paper, we give an analysis of PEP for V-OFDM. Assuming that maximum likelihood (ML) detection is applied at the receiver with perfect channel state information, the PEP conditioned on $H_l$ is written as

$$
\Pr (S_{lc} \rightarrow S_{le} | H_l) \leq \exp \left( \frac{d^2 (R_{lc}, R_{le})}{4N_0} \right).
$$

(9)

Here $d^2 (R_{lc}, R_{le} | H_l)$ is the squared Euclidean distance of two $M$-dim received signal vectors $R_{lc}$ and $R_{le}$, corresponding to the two transmit vectors $S_{lc}$ and $S_{le}$ as $R_{lc} = H_l S_{lc}$, $R_{le} = H_l S_{le}$ in the absence of noise. Thus, this distance is written as

$$
d^2 (R_{lc}, R_{le} | H_l) = \| H_l (S_{lc} - S_{le}) \|^2_2.
$$

(10)

Substituting (5) into (10) and using the property of unitary matrix, we have that

$$
d^2 (R_{lc}, R_{le} | H_l) = \| \hat{H}_l U_l (S_{lc} - S_{le}) \|^2_2.
$$

(11)
The squared Euclidean distance remains invariant to the unitary matrix \( U_l^H \). Following the same procedure in [21], in view of the relationship between V-OFDM and OFDM in (1) and (8), since \( \hat{H}_l \) is correlated with \( \{H_k\}_{k=0}^{N-1} \); further, \( \{H_k\}_{k=0}^{N-1} \) is an \( N \)-point FFT of zero padding of the channel vector \( h = [h_0, h_1, \ldots, h_{G-1}]^T \), then this distance can be substituted as

\[
d^2 (R_i^c, R_l^c | H_l) = \| \text{diag} (Q_l h) U_l (S_l^c - S_l^e) \|^2_2
\]

in which \( Q_l \in \mathbb{C}^{M \times G} \) is a partial and permutation matrix, constructed by extracting \( G \) leading columns and \( M \) non-consecutive rows \( \{l, l + L, \ldots, l + (M - 1) L\} \) with equidistant \( L \) of a \( N \)-point FFT matrix \( Q \), namely \( [Q_l]_{m,g} = \exp (-j2\pi (l + mL) g/N) \). Next, we define error vector \( e_l = S_l^c - S_l^e \) and create diagonal matrix \( E_{e,l} \in \mathbb{C}^{M \times M} \) as

\[
E_{e,l} = \text{diag} (U_l e_l) = \text{diag} (u_{l,1}^T e_l, \ldots, u_{l,M}^T e_l)
\]

where we denote the positive semidefinite Hermitian matrix \( T_{e,l} \in \mathbb{C}^{G \times G} \) as

\[
T_{e,l} = (E_{e,l} Q_l)^H (E_{e,l} Q_l)
\]

for presentation convenience. Thus, the inequality (9) can be rewritten using (14) as

\[
\Pr (S_l^c \rightarrow S_l^e | H_l) = \Pr (S_l^c \rightarrow S_l^e | h) \\
\leq \exp \left( \frac{h^H T_{e,l} h}{4N_0} \right).
\]

Finally, at high SNR, the error probability \( \Pr (S_l^c \rightarrow S_l^e) \) after averaging over the random channel vector \( h \) can be upper bounded as

\[
\Pr (S_l^c \rightarrow S_l^e) \leq \left[ C_{e,l} \frac{1}{4N_0} \right]^{-D_{e,l}}.
\]
Here, $D_{e,l} = \text{rank}(T_{e,l})$ and it characterizes the order of diversity, $C_{e,l} = \left(\prod_{d=0}^{\text{rank}(T_{e,l})-1} \lambda_d\right)^{1/D_{e,l}}$ and it represents the coding gain with $\{\lambda_d\}_{d=0}^{\text{rank}(T_{e,l})-1}$ being the nonzero eigenvalues of $T_{e,l}$.

From (17), the parameter $D_{e,l}$ determines how fast the average PEP decreases as the SNR increases, whereas $C_{e,l}$ determines the shift of this PEP curve in the SNR relative to a benchmark error rate curve of $(1/4N_0)^{-D_{e,l}}$.

**B. The Diversity Gain and Coding Gain**

Following the previous analysis, we define two important parameters, diversity gain $D_l$ and coding gain $C_l$, for each vector block $l \in \{0, 1, \cdots, L-1\}$ in V-OFDM system as follows:

- The diversity gain
  \[
  D_l = \min_{\forall a_{e_{l}} \neq 0, e_l \in \Psi} D_{e,l} = \min_{\forall a_{e_{l}} \neq 0, e_l \in \Psi} \text{rank}(T_{e,l}).
  \]  
  \[
  C_l = \min_{\forall a_{e_{l}} \neq 0, e_l \in \Psi} C_{e,l} = \min_{\forall a_{e_{l}} \neq 0, e_l \in \Psi} \left(\prod_{d=0}^{\text{rank}(T_{e,l})-1} \lambda_d\right)^{1/D_{e,l}}.
  \]

Based on the above definition, the BER performance of V-OFDM is highly determined by both the diversity gain $D_l$ and the coding gain $C_l$ at high SNR. In the following, the diversity gain $D_l$ and coding gain $C_l$ are studied in more details. Meanwhile, for analytical convenience, the coding gain will not be considered until the maximum diversity gain has been achieved.

Examining (15), it is shown that the rank of $T_{e,l} \in \mathbb{C}^{G \times G}$ is related to the length of vector, $M$, and is limited by the number of channel taps, $G$. Therefore, three cases: $M < G$, $M = G$, and $M > G$ shall be discussed, respectively.

Case 1) For the case of $M < G$, consider a matrix $A$, by employing the matrix property $\text{rank}(A^H A) = \text{rank}(A)$, it is shown that

\[
\text{rank}(T_{e,l}) = \text{rank}(E_{e,l} Q_l).
\]

It is well known that FFT matrix is an orthogonal and Vandermonde matrix, implying that its determinant can be rapidly computed and cannot be zero. Namely, its column or row vectors are linearly independent. Then note that $Q_l \in \mathbb{C}^{M \times G}$ is a submatrix permuted from an $N \times N$ FFT matrix. Further, if we again construct a submatrix $\bar{Q}_l \in \mathbb{C}^{M \times M}$, $[\bar{Q}_l]_{m,s} = \exp(-j2\pi(l + mL)s/N)$ by extracting the leading $M$ columns of $Q_l$, then it is easily observed that $\bar{Q}_l$ is also a Vandermonde matrix with non-zero determinant.
That is, the row vectors of \( \bar{Q}_l \) are also linearly independent. By Linear Algebra, if a group of row vectors are linearly independent, then adding some elements to the end of the vectors, they are still linearly independent. Hence, \( \text{rank} (Q_l) = M \). Here, bringing up a matrix \( C_l = Q_l^H (Q_l Q_l^H)^{-1} \), it would result in \( Q_l C_l = I_M \). Further, for a matrix pair \( A \) and \( B \), we have \( \text{rank} (AB) \leq \min \{ \text{rank} (A), \text{rank} (B) \} \). It then follows that

\[
\text{rank} \left( E_{e,l} I_M \right) = \text{rank} \left( E_{e,l} Q_l C_l \right) \\
\leq \text{rank} \left( E_{e,l} Q_l \right) \leq \text{rank} \left( E_{e,l} \right). \tag{21}
\]

As a result

\[
\text{rank} \left( T_{e,l} \right) = \text{rank} \left( E_{e,l} Q_l \right) = \text{rank} \left( E_{e,l} \right). \tag{22}
\]

Recall that \( E_{e,l} \) is an \( M \times M \) diagonal matrix. In view of that, the diversity gain \( D_l = \min_{\forall e_l \neq 0, e_l \in \Psi} \text{rank} \left( T_{e,l} \right) \) depends on the lowest number of nonzero diagonal elements in \( E_{e,l} \) for all \( e_l \neq 0, e_l \in \Psi \). In other words, the diversity gain \( D_l \) is at most \( M \) and the upper bound \( M \) can be achieved if and only if

\[
\Delta_l = \min_{e_l \neq 0, e_l \in \Psi} | \det (E_{e,l}) | \\
= \min_{e_l \neq 0, e_l \in \Psi} \prod_{i=1}^{M} | u_{l,i}^T e_l | > 0. \tag{23}
\]

Therefore, for V-OFDM, in case of \( M < G \), the diversity gain of each vector block, is uniquely determined by the diagonal matrix \( E_{e,l} \), equivalently, \( \Delta_l \).

Case 2) For the case of \( M = G \), the permutation matrix \( Q_l \) has been adjusted to a \( G \times G \) invertible matrix. Followed by similar analysis as in Case 1), the rank of \( T_{e,l} \) is computed as

\[
\text{rank} \left( T_{e,l} \right) = \text{rank} \left( E_{e,l} Q_l \right) = \text{rank} \left( E_{e,l} \right). \tag{24}
\]

As \( T_{e,l} \) is a \( G \times G \) matrix with maximum possible rank of \( G \), the maximum diversity gain \( D_l \) is at most \( G \). Hence, VB\( l \) will collect the maximum diversity gain \( M = G \) as long as \( \Delta_l > 0 \) holds. On the other hand, following the definition of the coding gain \( C_l \) in (19), in case of \( M = G \), it is then shown that \( C_l \)
can be expressed as

\[ C_l = \min_{\forall e_1 \neq 0, e_2 \in \Psi} \left[ \det (T_{e,l}) \right]^{1/G} \]
\[ = \left[ \det \left( Q_l^H Q_l \right) \right]^{1/G} \left( \Delta_l^2 \right)^{1/G}. \]  

(25)

From (25), \( C_l \) is linked to two parameters, \( \det \left( Q_l^H Q_l \right) \) and \( \Delta_l \). Because \( [Q_l]_{m,g} = \exp \left( -j2\pi (l + mL) g/N \right) \), then when \( M \geq G \), the elements of \( Q_l^H Q_l \) can be directly calculated as

\[ \sum_{m=0}^{M-1} \exp \left( -\frac{j2\pi (l + mL) g}{N} \right) \exp \left( \frac{j2\pi (l + mL) g'}{N} \right) \]
\[ = \exp \left( -\frac{j2\pi l (g - g')}{N} \right) \sum_{m=0}^{M-1} \exp \left( -\frac{j2\pi m (g - g')}{M} \right) \]
\[ = \begin{cases} M & g = g' \\ 0 & g \neq g' \end{cases} \]

(26)

and we have \( Q_l^H Q_l = M I_G \). When \( M < G \), it has \( Q_l^H Q_l \neq M I_G \). The upper bound of \( \Delta_l \) is given by [23] as (note this upper bound exists for any matrix with power limitation in [23]. Certainly, it is also established for unitary matrix \( U_l \))

\[ \Delta_l \leq \Delta_{\max} = \left( d_{\Phi,\min}/\sqrt{M} \right)^M \]

(27)

in which \( d_{\Phi,\min} = \min \{|S_1 - S_2|, S_1, S_2 \in \Phi, S_1 \neq S_2\} \), the minimum distance among all points in a two-dimensional constellation \( \Phi \). According to [23], \( \Delta_{\max} \) can be reached by some algebraic construction of \( U_l \). As a result, since \( \Delta_l \leq \Delta_{\max} \), then equation (25) is able to be simplified as

\[ C_l = G \left( \Delta_l^2 \right)^{1/G} \leq C_{\max} = d_{\Phi,\min}^2 \]

(28)

Here, only when \( \Delta_l = \Delta_{\max} \), maximum coding gain \( C_{\max} = d_{\Phi,\min}^2 \) can be reached. Otherwise, the coding gain will always be less than \( d_{\Phi,\min}^2 \). Based on the above analysis, the coding gain for each VB is still closely determined by the specific value of \( \Delta_l \) if \( \Delta_l > 0 \). To this end, let us examine the case of \( M > G \) for illustrative completeness.

Case 3) For the case of \( M > G \), we still analyze the diversity gain by examining the rank of \( T_{e,l} \).

Following the similar process as in the previous section, if \( \Delta_l > 0 \), it means that \( E_{e,l} \) is an invertible matrix and thus \( \text{rank} \left( T_{e,l} \right) = \text{rank} \left( E_{e,l} Q_l \right) = \text{rank} \left( Q_l \right) \). As \( Q_l^H Q_l = M I_G \) by (26), we have \( \text{rank}(Q_l) = \)
rank \((I_G) = G\). It means that rank \((T_e,l) = G\) and the maximum diversity gain \(D_l = G\) for VBs are also attained. Similarly, if rank \((T_e,l) = G\) is established, we further examine the coding gain \(C_l\). Since \(Q_l\) is not a square matrix, \(\det (T_e,l)\) cannot be directly calculated as (25). However, after direct computation, it is not difficult to find that all the diagonal entries of \(T_e,l\) equal to \(\text{Tr} (E_{e,l}^H E_{e,l})\). It is found that \(\text{Tr} (E_{e,l}^H E_{e,l}) = \|U_l e_l\|^2 = (e_l)^H (e_l)\). Then, using Hadamard inequality [31], we have \(\det (T_e,l) \leq \left(\text{Tr} (E_{e,l}^H E_{e,l})\right)^G = \left((e_l)^H (e_l)\right)^G\). Thus,

\[
C_l = \min_{\forall e_l \neq 0, e_l \in \Psi} \det (T_e,l)^{1/G} \leq \min_{\forall e_l \neq 0, e_l \in \Psi} \left((e_l)^H (e_l)\right) \bigg|_{e_l = \phi_{\text{min}}} = d_{\phi_{\text{min}}}^2
\]  

(29)
in which \(g\) is an arbitrary column of the identity matrix, i.e., \(g = [1, 0, \cdots, 0]^T\).

The upper bound of the coding gain \(C_l\) is still \(d_{\phi_{\text{min}}}^2\), thereby we draw the conclusion that \(d_{\phi_{\text{min}}}^2\) must be the maximum coding gain for V-OFDM system when the maximum diversity gain is achieved. As in Case 2), we know that the maximum coding gain \(C_l = d_{\phi_{\text{min}}}^2\) can be achieved if and only if \(\Delta_l = \Delta_{\text{max}}\). In Case 3), one may naturally ask what condition must be satisfied so that the maximum coding gain \(C_l = d_{\phi_{\text{min}}}^2\) can be reached and whether \(\Delta_l = \Delta_{\text{max}}\) is still a sufficient and necessary condition. In fact, because \(Q_l\) is not a square matrix, the analysis of \(C_l\) is very difficult. However, we can generally induce that \(\Delta_l = \Delta_{\text{max}}\) is still a sufficient condition but may no longer be a necessary one. We shall explain this in details in the next subsection.

Based on the analytical results in the above three cases, we conclude that the diversity gain and the coding gain of VB\(_l\) of V-OFDM are completely determined by the corresponding \(\Delta_l\). Specifically, if \(\Delta_l > 0\), this vector block of V-OFDM can collect the diversity gain of \(M\), at most \(G\). While \(\Delta_l = 0\), \(M \leq G\), the diversity gain is more specifically decided by the lowest number of nonzero diagonal elements in \(E_{e,l}\) for all \(e_l \neq 0, e_l \in \Psi\). Moreover, if \(\Delta_l\) is able to reach up to \(\Delta_{\text{max}}\), for \(M = G\), the upper bound of coding gain \(d_{\phi_{\text{min}}}^2\) in V-OFDM will be obtained by this VB.

In the following subsection, we shall just focus on the analysis of \(\{\Delta_l\}_{l=0}^{L-1}\) which will be largely based on algebraic number theory.
C. The Analysis of $\Delta_l$

Since $\Phi$ is a two-dimensional constellation with a finite size $|\Phi|$, the alphabet set $\Psi$ to which $e_l$ belongs must have a finite size $|\Psi|$. It is thus possible to examine $\Delta_l$ by using exhaustive computer search. However, the exhaustive search will become prohibitive when the length of VB and modulation order are large (e.g. $M = 16$, $|\Phi| = 16$, so $|\Psi| = 16^{16} = 1.8447 \times 10^{19}$). It is therefore necessary to resort to a mathematical tool. As $\Delta_l = \min_{e_l \neq 0, e_l \in \Psi} \left| \det (E_{e_l}) \right|$ and $\Delta_l$ is decided by $U_l$ in (6), a further analysis of $U_l$ will be presented. Firstly, we will reveal the symmetrical characteristic of $\{\Delta_l\}^{L-1}_{l=1}$.

**Theorem 1:** $\{\Delta_l\}^{L-1}_{l=1}$ are symmetrical on the central point $\Delta_{L/2}$, namely

$$\Delta_l = \Delta_{L-l}, \text{ for } l = 1, 2, \cdots, L - 1. \quad (30)$$

**Proof:** The entries of $U_l$ can be written as $[U_l]_{s,m} = M^{-1/2} \exp \left( -j2\pi (l + sL) m/N \right)$ (c.f. (6)); accordingly, we have $[U_{L-l}]_{s,m} = M^{-1/2} \exp \left( -j2\pi (L - l + sL) m/N \right)$. As a result

$$[U_{L-l}]_{M-s-1,m} = \frac{1}{\sqrt{M}} \exp \left( -j \frac{2\pi (L - l + (M - s - 1)L) m}{N} \right)$$
$$= \frac{1}{\sqrt{M}} \exp \left( j \frac{2\pi (l + sL) m}{N} \right)$$
$$= [U_l]_{s,m}^*.$$ \quad (31)

which means that the $M - s - 1$-th row of $U_{L-l}$ is the conjugate form of the $s$-th row of $U_l$. Concerning with a conventional symmetrical constellation, it is then straightforward to show that the equation (30) holds.

The above symmetrical characteristic implies that the performance of $VB_l$ should be identical with that of $VB_{L-l}$, which conveys that only half of VBs can definitely represent the entire performance of all VBs based V-OFDM system. Furthermore, observing (6), for a fixed $l$, $U_l$ is further written as the following form

$$U_l = \frac{1}{\sqrt{M}} \begin{bmatrix} 1 & \alpha_{l,0} & \cdots & \alpha_{l,0}^{M-1} \\ 1 & \alpha_{l,1} & \cdots & \alpha_{l,2}^{M-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_{l,M-1} & \cdots & \alpha_{l,M-1}^{M-1} \end{bmatrix} \quad (32)$$

in which $\alpha_{l,m} = \exp \left( -j2\pi l/N \right) \exp \left( -j2\pi m/M \right)$, $m = 0, 1, \cdots, M - 1$. 

In previous achievements, a class of important Vandermonde/unitary matrix, $\Theta$, is proposed in [18], [23], [25] and constructed using algebraic number theory. To be specific,

$$
\Theta = \begin{bmatrix}
1 & \alpha_0 & \cdots & \alpha_0^{M-1}
1 & \alpha_1 & \cdots & \alpha_1^{M-1}
\vdots & \vdots & \ddots & \vdots \\
1 & \alpha_{M-1} & \cdots & \alpha_{M-1}^{M-1}
\end{bmatrix}_{M \times M}
$$

(33)

where $M$ is power of two and $\{\alpha_i\}_{i=0}^{M-1}$ are selected as $M$ roots of a special polynomial $x^M - j = 0$ with $\alpha_0 = \exp(j2\pi/4M)$, $\alpha_m = \alpha_0 \exp(j2\pi m/M)$, $m = 0, 1, \cdots, M - 1$ in order to realize the upper bound of $\Delta_{\max}$.

Furthermore, remarkable results, regarding systematic and optimal cyclotomic lattices and diagonal space-time block codes is presented in [26], [27]. Specifically, the rotated matrix $G_{m,n}$ is given by

$$
G_{m,n} = \begin{bmatrix}
\zeta_N & \zeta_2^N & \cdots & \zeta_{L_t}^N \\
\zeta_1^{n_2m} & \zeta_2^{1+n_2m} & \cdots & \zeta_{L_t}^{1+n_2m} \\
\vdots & \vdots & \ddots & \vdots \\
\zeta_1^{(1+n_Ltm)} & \zeta_2^{1+n_Ltm} & \cdots & \zeta_{L_t}^{1+n_Ltm}
\end{bmatrix}_{L_t \times L_t}
$$

(34)

where $\zeta_N = \exp(j2\pi/N)$, $0 \leq n_i < n$, $1 \leq i \leq L_t$, are $L_t$ distinct integers such that $1 + n_im$ and $M = mn$ are co-prime, and $L_t = \phi(N)/\phi(m)$. The lattice with generating matrix $G_{m,n}$ is a full diversity cyclotomic lattice and its optimality is proved in [26]. Also, in [28], the authors have summarized existing rotated matrices using algebraic number theory. However, a closer investigation towards [18], [23], [28] indicates that the unitary matrices $\{U_l\}_{l=0}^{L-1}$ in (32) are not covered by the existing results. For example, the majority of the optimal matrices $G_{m,n}$ (c.f. (34)) are not unitary.

The variation of $l$ from 0 to $L - 1$ makes the novel problem originated from V-OFDM become more general and quite interesting. The unknown characteristics of $\{U_l\}_{l=0}^{L-1}$ are desirable to be discovered. The most important task now is to examine whether $\{\Delta_l\}_{l=0}^{L-1}$ can be greater than zero, then we will testify whether some $\Delta_l$ can reach up to $\Delta_{\max}$. We need some definitions and facts of algebraic number theory as follows (reader are referred to [23] and [29] for detailed definitions)

(T1) (Minimal Polynomial): $m_{\alpha,\mathbb{F}}(x)$ denotes the minimal polynomial (a polynomial with lowest degree) of $\alpha$ over a field $\mathbb{F}$ with $\deg(m_{\alpha,\mathbb{F}}(x))$ denoting its degree.
(T2) (Algebraic extension over $\mathbb{Q}(j)$): Let $\alpha$ be a root of a nonzero polynomial $\in \mathbb{Q}(j)[x]$. The field extension $\mathbb{Q}(j)(\alpha)$ is called an algebraic extension of $\mathbb{Q}(j)$ with degree $[\mathbb{Q}(j)(\alpha) : \mathbb{Q}(j)]$ equal to $\deg (m_{\alpha,\mathbb{Q}(j)}(x))$.

(T3) (Cyclotomic Polynomial): If $P \in \mathbb{N}$, the $P$-th cyclotomic polynomial is defined as $\Phi_P(x) = \prod_{k \in \kappa} (x - \exp(-j2\pi k/P))$, where $\kappa := \{k : (k, N) = 1$ and $1 \leq k < P$ and $\phi(P)$ is its degree. More important, the coefficient of $\Phi_P(x)$ are integers, i.e., $\Phi_{15}(x) = x^8 - x^7 + x^5 - x^4 + x^3 - x + 1$.

(T4) (Extension of an embedding and relative norm of a field): See [23], [29] for detailed definition. A useful property is that if $\beta \in \mathbb{Q}(j)(\alpha)$ is integral over $\mathbb{Z}[j]$, then the relative norm of $\beta$ from $\mathbb{Q}(j)(\alpha) \in \mathbb{Z}[j]$.

(T5) The polynomial $\Phi_P(x)$ is the minimal polynomial $m_{\alpha,\mathbb{Q}}(x)$ of $\alpha = \exp(-j2\pi/P)$. $\Phi_P(\alpha^i) = 0$ for any $i \in \mathbb{Z}$ such that $\gcd(i, P) = 1$.

(T6) (A property of Euler’s totient function $\phi(P)$): If $P$ is prime, then $\phi(P^n) = P^{n-1}(P - 1)$. In this case, $\phi(2^n) = 2^{n-1}$.

Based on the definitions and facts above, the following theorem regarding $\Delta_l$ can be proved:

**Theorem 2:** Consider two classes of modulation constellation $\Phi$, one is any constellations carved from the lattice $\mathbb{Z}$, such as PAM or BPSK modulation, the other is any constellations carved from the lattice $\mathbb{Z}[j]$, such as all QAM (including QPSK) modulation. Note that general $M$-PSK modulation, i.e., 8PSK, are not contained in the two classes. Then, we have

- For VB$_0$ and any constellations carved from the lattice $\mathbb{Z}$ or $\mathbb{Z}[j]$, $\Delta_0 = 0$.
- For VB$_l$, $l \in \{1, L - 1\}$ and any constellations carved from the lattice $\mathbb{Z}$, $\{\Delta_l\}_{l=1}^{L-1}$ are greater than zero.
- For VB$_l$, $l \in \{1, L - 1\}$ but $l \neq L/2$, and any constellations carved from the lattice $\mathbb{Z}[j]$, $\{\Delta_l\}_{l=1, l \neq L/2}^{L-1}$ are greater than zero, where $\Delta_{L/2} = 0$.
- For any constellations carved from the lattice $\mathbb{Z}[j]$, $\Delta_{L/4}$ and $\Delta_{3L/4}$ can reach up to $\Delta_{\text{max}}$. In comparison, $\Delta_{L/4}$, $\Delta_{3L/4}$, and $\Delta_{L/2}$ all can obtain the upper bond $\Delta_{\text{max}}$ for any constellations carved from the lattice $\mathbb{Z}$.

**Proof:** For a given $l$, recall

$$E_{\nu,l} = \text{diag} \left( U_{\nu} e_{l} \right) = \text{diag} \left( u_{\nu,0}^{T} e_{l}, \ldots, u_{\nu,M-1}^{T} e_{l} \right)$$ (35)
and $u_{l,m}^T = M^{-1/2} \left[ 1, \alpha_{l,m}, \ldots, \alpha_{l,m}^{M-1} \right]$. For any constellations carved from the lattice $\mathbb{Z}$, namely, $d_{\phi, \min}^{-1} \in \mathbb{Z}^M$ and for all $m \in \{0, 1, \ldots, M-1\}$, if the following condition is satisfied

$$[Q(\alpha_{l,m}) : Q] \geq M,$$  \hspace{1cm} (36)

or equivalently, the degree of minimal polynomial of $\alpha_{l,m}$ over $\mathbb{Z}$, $\text{deg} \left( m_{\alpha_{l,m},Q}(x) \right)$, is more than or equal to $M$. Then we can confirm that $u_{l,m}^T e_l \neq 0$ with $e_l \neq 0$. It should be noticed that the condition in (36) is a sufficient condition. If (36) is not supported, we cannot make a decision that $u_{l,m}^T e_l = 0$ with $e_l \neq 0$. As such, a further investigation is required. Similarly, for any constellations carved from the lattice $\mathbb{Z}[j]$, since $[Q(j) : \mathbb{Q}] = 2$ and $N = 2^{m_0}$, $M = 2^{m_0}$, we need

$$[Q(\alpha_{l,m}) : \mathbb{Q}] \geq 2M$$  \hspace{1cm} (37)

for all $m \in \{0, 1, \ldots, M-1\}$ so that

$$[Q(j)(\alpha_{l,m}) : Q(j)] \geq M,$$  \hspace{1cm} (38)

i.e., the degree of minimal polynomial of $\alpha_{l,m}$ over $\mathbb{Z}[j]$, is more than or equal to $M$. To be continued, recall that $M = 2^{m_0}$ and $L = 2^{l_0}$ for some positive integers $m_0$ and $l_0$. Hence, we have $N = 2^{m_0+l_0}$ and thus we can write

$$\alpha_{l,m} = \exp \left( -j2\pi (l + 2^{l_0}m) / 2^{m_0+l_0} \right)$$  \hspace{1cm} (39)

with $m \in \{0, 1, \ldots, 2^{m_0} - 1\}$. Now first consider two special cases, $\Delta_0$ and $\Delta_{L/2}$. For $l = 0$, it has $\alpha_{0,m} = \exp \left( -j2\pi m / 2^{m_0} \right)$ and thus $[Q(\alpha_{0,m}) : \mathbb{Q}] \leq \phi(2^{m_0}) = M/2$, which is far away from (36). Then for $l = L/2 = 2^{l_0-1}$, it has $\alpha_{L/2,m} = \exp \left( -j2\pi (1 + 2m) / 2^{m_0+1} \right)$; because $1 + 2m$ is odd, then $\gcd(1 + 2m, 2^{m_0}) = 1$, that is, $[Q(\alpha_{L/2,m}) : \mathbb{Q}] = \phi(2^{m_0+1}) = M$. Clearly, just (36) is satisfied.

As mentioned above, $\Delta_0$ under constellations carved from $\mathbb{Z}$ and $\mathbb{Z}[j]$, and $\Delta_{L/2}$ under constellations carved from $\mathbb{Z}[j]$, need to be further investigated later. For other $\Delta_l$, assuming $L > 2$, namely, $l_0 \geq 2$ (otherwise, we have $L/2 = L - 1$), by similar analysis, for $l \in \{1, 2, \ldots, L-1\}$ but $l \neq L/2$, we have $[Q(\alpha_{l,m}) : \mathbb{Q}] \geq \phi(2^{m_0+2}) = 2M$. In other words, the degree of the minimal polynomial of $\alpha_{l,m}$ over $\mathbb{Z}$ is at least $2M$ and over $\mathbb{Z}[j]$ is at least $M$. Thus, $u_{l,m}^T e_l \neq 0$ with $e_l \neq 0$, and $\Delta_l > 0$. Partial conclusions in Theorem 2 are now established.
Next, we will further investigate $\Delta_0$ and $\Delta_{L/2}$. Firstly, for $l = 0$, let vector $z \in \mathbb{C}^{M \times 1}$ denote an all-one vector. The first row of $U_0$, $u_{0,0}^T$, equals $M^{-1/2}z^T$. Now, consider a particular pair $\{S_0^e, S_0^f\}$ with error vector $e_0^f = S_0^f - S_0^e = d_{\Phi, \min}z$. Note this error vector would happen for any constellations carved from $\mathbb{Z}$ or $\mathbb{Z}[j]$. Because $\{u_{0,m}^T\}_{m=0}^{M-1}$ are mutually orthogonal, we have

$$E_{e,0} = \text{diag} \left( u_{0,0}^T e_0^f, \ldots, u_{0,M-1}^T e_0^f \right)$$

$$= \text{diag} \left( d_{\Phi, \min} M^{-1/2} z^T z, 0, 0, \ldots 0 \right)$$

$$= \text{diag} \left( \sqrt{M} d_{\Phi, \min}, 0, 0, \ldots 0 \right).$$

(40)

Obviously, the lowest number of nonzero diagonal elements in $E_{e,0}$ for all $e_0 \neq 0$, $e_0 \in \Psi$, equals one. Naturally, $\Delta_0 = 0$. Secondly, for $l = L/2$ and $m = 0, 2, 4, \ldots, M-2$, it has $\alpha_{L/2, m}^{M/2} = \exp \left( -j \pi \times 2^{m-1} / 2^m + 1 \right) = j$. It then means $u_{L/2, m}^T = M^{-1/2} \left[ \frac{1, \ldots, j, \ldots}{M/2} \right]^T$. For any constellations carved from $\mathbb{Z}[j]$, consider a particular pair $\{S_{L/2}^e, S_{L/2}^f\}$ with error vector $e_{L/2}^f = S_{L/2}^f - S_{L/2}^e = d_{\Phi, \min} \left[ \frac{1, 0, \ldots, 0, j, 0, \ldots, 0}{M/2} \right]^T$, we have $u_{L/2, m}^T e_{L/2}^f = 0$. Resultingly, the lowest number of nonzero diagonal elements in $E_{e, L/2}$ for all $e_{L/2} \neq 0$, $e_{L/2} \in \Psi$, is at most $M/2$ and also $\Delta_{L/2} = 0$.

Lastly, we shall examine whether some $\{\Delta_l\}$ can reach up to $\Delta_{\max}$. We find, particularly for $l = 3L/4$, $\alpha_{3L/4, m}^{M/4} = \exp \left( -j 2\pi \times (3 + 4m) / 4 \right) = \exp \left( j 2\pi \times (1 + 4m) / 4 \right)$. Here, $\{\alpha_{3L/4, m}^{M/4}\}_{m=0}^{M-1}$ are precisely the values $\{\alpha_m\}_{m=0}^{M-1}$ of $\Theta$ in (33). This is not a new finding and previous work in [18], [23] and [25] has proved the value of $\Delta_{3L/4} = \Delta_{\Theta}$ can achieve $\Delta_{\max}$. Interestingly, V-OFDM just contains this case by coincidence. The main reason for $\Delta_{3L/4} = \Delta_{\max}$ can be explained as follows. For $l = 3L/4$, $\{\alpha_{3L/4, m}^{M-1}\}_{m=0}^{M-1}$ are just all the roots of $x^{M} - j = 0$, a minimal polynomial over $\mathbb{Z}[j]$ with degree of $M$. Let $d_{\Phi, \min} e_l \in \mathbb{Z}[j]^M$. The first diagonal elements of $E_{e, 3L/4}$ is $u_{3L/4, 0}^T e_l$; other diagonal elements, $\{u_{3L/4, m}^T e_l\}_{m=1}^{M-1}$, are the isomorphisms and images of $u_{3L/4, 0}^T e_l$. Then

$$(M^{1/2} d_{\Phi, \min}^{-1})^M \det \left( E_{e, 3L/4} \right) = \prod_{m=0}^{M-1} \left( M^{1/2} u_{3L/4, 0}^T \right) (d_{\Phi, \min}^{-1} e_l)$$

(41)

coincides with the definition of relative norm of a field (c.f. (T4), please see [23] for details). Using the property of relative norm of a field, we have

$$(M^{1/2} d_{\Phi, \min}^{-1})^M \det \left( E_{e, 3L/4} \right) \in \mathbb{Z}[j] \setminus 0$$

(42)
and thus
\[
\left| \left( M^{1/2} d_{\Phi, \text{min}}^{-1} \right)^M \det (E_{c, 3L/4}) \right| \geq 1,
\]
leading to \( \left| \det (E_{c, 3L/4}) \right| \geq \left( d_{\Phi, \text{min}} / \sqrt{M} \right)^M = \Delta_{\text{max}} \). Together with (27), we know that \( \Delta_{3L/4} = \Delta_{\text{max}} \).

Moreover, using Theorem 2, for \( l = L/4 \), because of the property of symmetry, we can directly obtain our novel finding, \( \Delta_{L/4} = \Delta_{\text{max}} \). In fact, it can be found that \( \{ \alpha_{L/4, m} \}_{m=0}^{M-1} \) now are just all the roots of another minimal polynomial, \( x^M + j = 0 \). With the similar analysis as \( l = 3L/4 \), we also arrive at \( \Delta_{L/4} = \Delta_{\text{max}} \). To this end, for \( l = L/2 \) and constellations only carved from \( \mathbb{Z} \), \( \{ \alpha_{L/2, m} \}_{m=0}^{M-1} \) are the roots of \( x^M + 1 = 0 \), a minimal polynomial over \( \mathbb{Z} \) with degree of \( M \). Likewise, \( \Delta_{L/2} = \Delta_{\text{max}} \). We have completed the proof of all the conclusions in Theorem 2.

Remarks: In the proof of Theorem 2, we have assumed \( L = 2^{l_0} \) for effective FFT implementation and \( N = 2^{n_0} \) for conventional arrangement for the number of subcarriers in OFDM. Thus, \( M \) is also power of two. This assumption is consistent with the literature towards V-OFDM. If we only restrict \( L = 2^{l_0} \) but \( M \) is no longer power of two, i.e., \( M = 3 \), then this problem can also be resolved based on minimal polynomial but more vector blocks cannot make \( \Delta_l > 0 \) (For \( M = 3 \), there are three vector blocks). Meanwhile, \( \Delta_l \) cannot achieve \( \Delta_{\text{max}} \) in general. Furthermore, the problem with arbitrary integer \( M, L, N \), becomes more complex and a classification of integer (odd, even, etc.) is needed.

In Theorem 2, we have shown that some values of \( \{ \Delta_l \} \) can reach up to \( \Delta_{\text{max}} \). Then, one will naturally ask what is the distribution of \( \{ \Delta_l \} \). In fact, \( \{ \alpha_{l, m} \}_{m=0}^{M-1} \) may belong to a high order minimal polynomial or different minimal polynomials; besides, \( M, L \) and \( N \) are all variable parameters. Determining the profile is intractable. For this reason, we shall resort to computer validation.

In Fig. 2, values of \( \{ \Delta_l \}_{l=0}^{L-1} \) are delineated for a fixed vector block size, \( M = 4 \), where a QPSK constellation (constellation carved from \( \mathbb{Z}[j] \)) and a total number of \( N = 256 \) are employed with \( L = 64 \). Several important observations can be made from this figure. Firstly, ignoring \( \Delta_0 \), it is seen that the remaining values of \( \{ \Delta_l \}_{l=1}^{L-1} \) are mirrored at the central point \( \Delta_{L/2} = 32 \). Namely, \( \Delta_l = \Delta_{L-l} \) for \( l \in \{ 1, 2, \cdots, L-1 \} \). This observation agrees with Theorem 1. Secondly, excluding \( \Delta_0 \) and \( \Delta_{L/2} = 32 \), we find that all \( \{ \Delta_l \}_{l=1, l \neq L/2}^{L-1} \) are greater than zero. Following the previous discussion, this property will play a vital role in the performance of V-OFDM under multi-path fading channel. Thirdly, for an energy normalized QPSK constellation with \( d_{\Phi, \text{min}} = \sqrt{2} \) and thus \( \Delta_{\text{max}} = \left( d_{\Phi, \text{min}} / \sqrt{M} \right)^M = 0.25 \), we find \( \Delta_{L/4 = 16} = \Delta_{3L/4 = 48} = \Delta_{\text{max}} \), which are conformed to previous achievements [23] and Theorem 2. Lastly, one can observe significant fluctuation associated with \( \{ \Delta_l \}_{l=0}^{L-1} \). Further, even though \( \Delta_1 = \Delta_{63} > 0 \), but
\[ \Delta_1 = \Delta_{63} = 2.363e^{-4}, \] this value will inevitably lead to a loss in coding gain. The main reason here is that \( \{\alpha_{1,m}\}_{m=0}^{M-1} \) belong to a high order minimal polynomial.

Fig. 3 illustrates similar results as Fig. 2 but a 4-PAM constellation (constellation only carved from \( \mathbb{Z} \)) is employed. We also choose \( M = 4, N = 256 \) with \( L = 64 \). In this scenario, \( \Delta_{\text{max}} = 0.04 \). We find \( \Delta_{L/4=16} = \Delta_{L/2=32} = \Delta_{3L/4=48} = \Delta_{\text{max}} \). Under this configuration, it is interesting that some other \( \{\Delta_i\} \) also equal \( \Delta_{\text{max}} \), which implies that we have found more matrices \( \{U_l\} \) achieving the maximum coding gain through analysis of V-OFDM and these matrices are not available in the previous works like [18], [23], [25] and [26]. This is a new finding in the construction of cyclotomic matrices. Other observations are similar with Fig. 2.

D. Summary

In this subsection, we draw some important conclusions regarding the performance of V-OFDM under ML detection at the receiver based on the analytical findings in the previous subsections.

When \( M < G \), for any constellations carved from \( \mathbb{Z}[j] \), almost all the VBs (excluding \( \text{VB}_0 \) and \( \text{VB}_{L/2} \)) in V-OFDM are able to attain the diversity gain of \( M \). For analytical convenience, we do not discuss the coding gain in this scenario. In fact, according to simulation confirmation in the Section VI, we find the coding gain for each VB is also determined by the amplitude of \( \Delta_i \), leading to varied BER performance for different VB. Further, these VBs perform the maximum diversity gain of \( G \) when \( M \geq G \) with corresponding coding gain of \( C_l \leq d_{\Phi,\text{min}}^2 \). In theory, \( \text{VB}_0, \text{VB}_{L/2} \) will only achieve the diversity gain of one and at most \( M/2 \), respectively. When \( M \) just equals \( G \), \( \text{VB}_{L/4}, \text{VB}_{3L/4} \) can obtain the maximum coding gain of \( d_{\Phi,\text{min}}^2 \) for any constellations carved from \( \mathbb{Z}[j] \) and \( \text{VB}_{L/4}, \text{VB}_{L/2}, \text{VB}_{3L/4} \) can obtain the maximum coding gain of \( d_{\Phi,\text{min}}^2 \) for any constellations carved from \( \mathbb{Z} \). For some specific \( M \) and constellations, some VBs would also achieve the maximum coding gain of \( d_{\Phi,\text{min}}^2 \) in terms of numerical results.

In particular, when \( M > G \), we find that quite a few of VBs harvest the maximum coding gain of \( d_{\Phi,\text{min}}^2 \). In Fig. 4, for \( M = 4, G = 4 \) or \( M = 2, G = 2 \), there are only few VBs which can contact with this target. However, given \( M = 4, G = 2 \), it can be found that significant increase of the coding gain is introduced and the maximum coding gain has been achieved by a large number of VBs. In Fig. 4, another interesting observation is, although \( C_{58} > C_{36} \) for \( M = 4, G = 4, C_{36} \) takes the lead in attaining \( d_{\Phi,\text{min}}^2 \) when \( G \) reduces from 4 to 2. Meanwhile, \( \Delta_i = \Delta_{\text{max}} \) becomes a sufficient condition rather than a necessary one. In general, when \( M \) exceeds the number of channel taps, further increasing \( M \) cannot
increase the diversity gain, but it exerts notable impact on the enhancement of coding gain. Nevertheless, a rigorous analysis on the phenomenon \( C_{58} > C_{36} \) but \( C_{36} \) takes the lead in attaining \( d_{\phi,\text{min}}^2 \) and the determinant calculation for a product of two non-square matrices are intractable. We shall leave them for future work.

IV. COMPARISONS AND DISCUSSIONS

In this section, we present some discussions on V-OFDM in comparison with other OFDM-related technologies, including channel-coded OFDM, constellation-rotated OFDM and precoded OFDM. Both characteristics and computational complexity will be compared.

A. Comparisons with Coded OFDM

By our analysis in Section III, now it is clear that the specific construction of \( U_l \) in (6) as well as the sample form of \( \hat{H}_l \) in (8) ensure that V-OFDM can obtain signal space diversity under multi-path fading channel with sufficient channel orders. As we know, signal space diversity is power- and bandwidth-efficient in contrast to time, frequency and spatial diversity, since it does not require redundancy and neither consumes extra resources in power and/or bandwidth. On the other hand, in order to collect frequency diversity for conventional OFDM systems, appropriate frequency interleaving and coding is necessary. Clearly, some powerful error correcting codes like convolutional codes and Turbo coding codes are able to boost the system performance significantly. However, attached redundancy, decoding complexity and delay are also introduced. Thus, we conclude that signal space diversity is an economical alternative to conventional frequency domain forward error control coding. Nevertheless, these two methods are not opposite for each other. A more advisable scheme is to combine V-OFDM with frequency channel coding so as to further exploit system diversity. In the following section, we will numerically compare coded OFDM with coded V-OFDM via a basic and an advanced combinations of signal space diversity and frequency coding diversity. In the consideration of computational complexity, since the number of complex multiplications of an \( L \)-point FFT is \( L \log_2 L \), OFDM requires \( 2N \log_2 N \) complex multiplications while V-OFDM requires \( 2M L \log_2 L = 2N \log_2 L \) complex multiplication for transceiver, which is relatively lower than OFDM.
B. Comparison with Constellation-rotated OFDM

More recently, a constellation-rotated V-OFDM (CRV-OFDM) is proposed in [19] to optimize the BER performance of V-OFDM. The main idea behind this scheme dates from (5) and (7). Since each VB corresponds to a varied $U_l$, leading to analytical difficulty and also different performance for each VB. As a result, in [19], the authors adopt a skillful and effective approach to eliminate this variability, namely, resorting to constellation-rotation technique. We give a brief introduction about the main procedure for CRV-OFDM. The first step is to construct a diagonal matrix $\Theta = \text{diag}(1, e^{j\theta}, \ldots, e^{j(M-1)\theta})$, where $\theta$ is the rotational angle for further design. As $U_l = F_M A_l$, the following task is to eliminate $A_l$, thus we have equivalent rotated matrix for each VB:

$$\Omega_l = U_l A_l^H \Theta = F_M A_l A_l^H \Theta = F_M \Theta.$$  (44)

$\Omega_l$ herein is only correlated with the rotation angle $\theta$. When $\theta$ is determined, the equivalent rotated matrix for each VB is identical. The optimization of $\theta$ depends on computer search. By exhaustive search, the optimal angles found in [19], for BPSK, are $\theta = \frac{\pi}{2}$, $\frac{15\pi}{128}$ for $M = 2, 4$, respectively, and, for QPSK, the optimal angles are $\theta = \frac{23\pi}{128}$ and $\theta = \frac{\pi}{8}$. Exhaustive search needs high complexity, but it can be done offline, thus this method is valid in practical system design for small $M$. The computational complexity for CRV-OFDM is slightly higher than V-OFDM. The total number of complex multiplications (not including ML decoding) for transceiver should be $2N\log_2 L + 2LM = 2N\log_2 L + 2N$.

C. Comparison with precoded OFDM

The precoded OFDM [21], [22] is also a consideration for channel null and symbol detectability with conventional OFDM. In fact, the idea of using linear precoding scheme is also related to that of signal space diversity. Here, we will give a quick review of linear constellation precoded OFDM proposed in [22]. For precoded OFDM, all subcarriers are partitioned into $L$ groups. A linear precoding matrix $\Theta$ is applied to each group. In the end, an IFFT operation changes the all precoded subcarriers from frequency domain to time domain. Interestingly, after our examination, under the specific groups division, precoded OFDM shows the similar input-output form while the main difference also comes from the rotated matrix. In particular, only when $M$ is the power of two, the rotated matrix of precoded OFDM is the same as that of $\text{VB}_{3L/4}$ in V-OFDM by coincidence. However, it should be mentioned here that, in practical view, precoded
OFDM is different from V-OFDM and CRV-OFDM because the subcarriers of the former is orthogonal while the latters are not. Their PAPR property, sensitivity to CFO, the channel estimation method as well as synchronization, should be different from each other, requiring further analytical study. We will leave this issue for our future work. For precoded OFDM, the total number of complex multiplications for transceiver are

\[
\frac{2N\log_2 N}{\text{IFFT/FFT}} + L M^2 + \frac{L M^2}{\text{before ML decoding}} = 2N\log_2 N + 2NM.
\]

V. SIMULATION RESULTS

In this section, simulation results are provided to confirm the analysis carried out in previous sections for V-OFDM systems. The QPSK modulation scheme with Gray mapping is employed. The multi-path channel is modeled as i.i.d. quasi-static Rayleigh fading channel. In our analysis, ML decoding at the receiver is required for extracting its diversity gain and coding gain; however, this algorithm is practically prohibitive to perform since the complexity depends exponentially on the length of VBs. For this reason, the sphere decoding is adopted in all the simulations for its lower computational complexity and near optimal performance [30].

In Fig. 5, we examine the achievable diversity gain of V-OFDM by illustrating the BER curves. A total of \( N = 256 \) subcarriers are considered. The length of VBs, \( M \), varies among 1, 2, 4, 8 and 16 with the FFT length of \( L \) adjusted to keep \( N = ML \). The multi-path channel has \( G = 20 \) channel taps and thus \( M < G \) is established. In the case with \( M = 1 \), the system corresponds to conventional OFDM system with FFT length of \( N \). We can find the uncoded OFDM has poor BER performance as its diversity gain is known to be one only. Further, it can be observed that the BER performance can be improved significantly when \( M \) increases. While \( M = 16 \), its performance under fading channel is close to the one under AWGN channel. But this comes at a cost for a high computational complexity. At this point, however, we cannot generally conclude that the maximum obtainable diversity gain for V-OFDM is equal to the length of VBs, since at the first glance, the diversity gain for V-OFDM seems to be less than 4 and 8 at high SNR. To explain this, we provide in Fig. 6 the BER performance of each individual VB in V-OFDM together with the overall BER performance when \( M = 4 \). From Fig. 6 at high SNR range (20-24dB), it can be readily observed that the diversity gain of V-OFDM is smaller than 4, which is due to the fact that some VBs (for example VB\(_0\)) fail to achieve the diversity order of 4; meanwhile, some VBs (for example VB\(_0\)) have quite small coding gain (as VB\(_1\), since \( \Delta_1 = 0.0002363 \ll 0.25 \) in Fig. 2), leading to serious degradation with BER performance even though they can obtain the diversity order.
of 4 in theory. For other VBs, as long as their coding gains are not too small, it is clear that their BER curves are almost parallel (as VB$_3$, VB$_4$, VB$_5$ and VB$_{16}$), demonstrating the diversity order of 4. Since the overall system performance is restricted by the VBs with the worst BER performance, it then arrives at the results shown in Fig. 5 that the diversity order of V-OFDM is slightly smaller than $M$.

Fig. 7 presents the similar simulated BER performance as Fig. 5, but with less number of channel taps $G = 2$. According to the previous analysis and the distribution of the coding gains in Fig. 4, when $M = 2$, in general, V-OFDM can collect the maximum diversity gain of two; however, it is clear that two VBs ($l = 0$ and $l = L/2$) fail to attain the maximum diversity gain and other VBs cannot all reach up to the maximum coding gain. Hence, some performance losses associated with V-OFDM are introduced. Here, it is known that increasing $M$ cannot elevate the diversity gain for most of VBs; but optimistically, quite a few VBs could obtain the maximum coding gain, which leads to a slight performance improvement. As a result, the performance limitation of V-OFDM is almost reached and there is no significant improvement for $M \geq 4$. This again confirms our analysis on the diversity and coding gain of V-OFDM. From the simulation result in this figure, choosing vector length equal to channel taps may provide a good trade-off between system performance and computational complexity for decoding.

In order to illustrate the diversity properties of V-OFDM more clearly, in Fig. 8, the performance of different VBs is compared to check the effects of the diversity gain and coding gain under the same system configuration. Here, another group of parameters, $M = 2$, $G = 2$, $N = 128$ are employed and we select VB$_0$, VB$_2$, VB$_6$, VB$_7$, VB$_{12}$, VB$_{L/2=32}$ and VB$_{3L/4=48}$ whose coding gain is conformed to Fig. 4. Clearly, since VB$_0$ and VB$_{L/2=32}$ fail to exploit the maximum diversity gain of two, they show the worst BER performance. In sharp comparison, VB$_{12}$ and VB$_{3L/4=48}$ can not only harvest the full diversity gain, but also successfully achieve the full coding gain. Accordingly, they show the best BER performance. The performance of other VBs, such as VB$_6$ and VB$_7$, lies between VB$_0$ and VB$_{3L/4=48}$.

Fig. 9 examines the symmetry of V-OFDM system obtained in Theorem 1 with the same system configuration as in Fig. 8. Obviously, VB$_l$ performs the same with VB$_{L-l}$. It should be mentioned here that two VBs with the same diversity gain and coding gain do not necessarily show identical BER performance. Other factors, such as kissing number [23], may cause slight differences. But for VB$_l$ and VB$_{L-l}$, as in the proof of Theorem 1, the $M - s - 1$-th row of $U_{L-l}$ is the conjugate form of the $s$-th row of $U_l$. So $U_{L-l}$ can be viewed as an image of $U_l$. In this case, under any system configuration, VB$_l$ must perform the same with VB$_{L-l}$. This is also another important property associated with V-OFDM.
systems.

In Fig. 10, we will further examine the performance of V-OFDM and OFDM when frequency-domain forward error correcting codes are applied in both systems. The number of subcarriers is \( N = 256 \) and the total channel taps \( G = 20 \). The rate 1/2 convolutional code with generator polynomials \( G_0 = 133_8 \) and \( G_1 = 171_8 \) as adopted in the IEEE Std. 802.11a is employed. The convolutional encoder is concatenated with a random interleaver to exploit frequency diversity. At the receiver, hard-decision based Viterbi decoder is used to decode convolutionally encoded data. For comparison, the performance of uncoded systems are also presented in Fig. 10. It is first observed that the coded OFDM effectively takes advantages of the diversity provided by the multi-path fading and provides significant performance improvement over the uncoded case. It is expected that the maximum diversity order collected by channel coding is at most \( G \). Second, V-OFDM can successfully exploit signal space diversity only by simple small-scale FFT implementation. According to using coding and signal space diversity inherited in V-OFDM as a constituent module, the system diversity can be sufficiently exploited. It is shown that simple cascade of channel coding and V-OFDM, is quite effective. More advanced cascaded scheme, V-OFDM based on iterative Turbo equalization (TE), which is detailed introduced in [15], could further exploit the system diversity.

In the end, in Fig. 11 and Fig. 12, we further compare V-OFDM with existing alternatives in terms of computational complexity and error performance. Since V-OFDM, CRV-OFDM and precoded OFDM all require ML decoder at the receiver, we only consider FFT in the complexity analysis, though it may not be fair for the uncoded OFDM system which does not involve ML decoding. From Fig. 11, it is shown that the lowest computational complexity for V-OFDM is achieved. In Fig. 12, a comparison between V-OFDM and existing alternatives in terms of error performance is characterized where we set \( N = 256 \), \( M = 2 \) and \( G = 2 \). Because of the difficulty in conducting ML (or SD) decoding algorithm for SC-FDE, this result is omitted. As we have illustrated in Fig. 8, differentiated performance for each VB introduces some degradation to V-OFDM. In contrast, as our statements in the Section IV, the BER performance of precoded OFDM should be identical with that of \( \text{VB}_{3L/4} \) under these parameters configuration, therefore it shows the best BER performance. The BER performance of CRV-OFDM is slightly poor than that of precoded OFDM, but the theoretical reason is unclear due to computer search of the rotation angle \( \theta \).
VI. Conclusion

In this paper, the performance limits of V-OFDM under multi-path fading channel in terms of diversity gain and coding gain are presented. We show the symmetrical characteristic of V-OFDM. By using algebraic number theory tools, we prove that the diversity gain associated with V-OFDM promotes with increasing $M$ but limited by total number of the channel taps. Some specific VBs, whose length equals the total number of channel taps, can not only harvest the maximum diversity gain but also achieve the maximum coding gain in V-OFDM system. Further increased $M$ exerts significant impact on the accessible coding gain for each VB. Numerical results illustrate that choosing the vector length equal to the number of channel taps strikes a good balance between the system performance and the computational complexity for detection. The analytical findings and conclusions in this work may offer insights into the design of practical V-OFDM based systems.

REFERENCES


## TABLE I
**NOTATIONS**

- **M**  
  The dimension (length) of vector blocks
- **L**  
  The size of IFFT and FFT operation
- **N**  
  The total number of subcarriers in V-OFDM, $N = M \times L$
- **P**  
  The number of cyclic prefix vectors
- **G**  
  The order of channel impulse
- **$F^{-1}_L$**  
  An $L$-point normalized IFFT matrix
- **$\Phi$**  
  A conventional two-dimension constellation of size $2^c$ (e.g., QAM)
- **$d_{\Phi,\text{min}}$**  
  The minimum distance among all points in $\Phi$
- **$\Psi$**  
  A limited set in which an error vector $e_l$ belongs to
- **$D_l$**  
  The diversity gain
- **$C_l$**  
  The coding gain

---

![Block Diagram of V-OFDM System](image-url)

**Fig. 1.** The block diagram of V-OFDM system.
Fig. 2. The distribution of $\Delta_l$ for each VB (QPSK, $M = 4$, $N = 256$).

Fig. 3. The distribution of $\Delta_l$ for each VB (4-PAM, $M = 4$, $N = 256$).
Fig. 4. The distribution of $C_l$ for each VB.

Fig. 5. BER performance of V-OFDM system ($N = 256, G = 20$).
Fig. 6. BER performance of V-OFDM system ($N = 256, G = 20$).

Fig. 7. BER performance of V-OFDM system ($N = 256, G = 2$).
Fig. 8. BER performance of different VBs in V-OFDM system ($N = 128$, $M = 2$, $G = 2$).

Fig. 9. BER performance of symmetrical VBs in V-OFDM system ($N = 256$, $M = 2$, $G = 2$).
Fig. 10. BER performance of coded OFDM and coded V-OFDM system ($N = 256, G = 20$).

Fig. 11. A comparison of computational complexity with existing alternatives.
Fig. 12. BER performances of V-OFDM and other alternatives ($N = 256, M = 2, G = 2$).