Capacity of Wireless Networks with Heterogeneous Node Distribution and Relationship

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ABSTRACT
This paper studies the throughput capacity of wireless networks with heterogeneous node distribution and relationship. We propose a simple model which captures the two key characteristics observed in such networks, i.e., small-worldness and power-law node degree distributions, and examine their impact on capacity. We show the fact that the heterogeneity of nodes’ geographical location leads to traffic locality and improves capacity. Moreover, multicasting may be employed to further enhance performance when information is desired to be published from the source to all its contacts, of which the number follows power-law distribution. In addition, we propose the corresponding capacity-achieving communication schemes which optimally exploit the underlying structure. Our study discloses how heterogeneity in terms of both node location and relationship may impact on network capacity from a theoretical perspective, and provides fundamental insight on the design and analysis of real wireless networks.

Keywords
Capacity, Wireless, Social Networks

1. INTRODUCTION
The structure of large scale wireless networks are nowadays reformed by a wide range of newly emerged and rapidly penetrating applications. Typical examples include online social networks such as MySpace, Facebook, Orkut, LiveJournal, Cyworld and Flickr, which have attracted tens of millions of users integrating these sites into their daily practices. These online social services have offered an unprecedented opportunity for measurement-based studies on human social networks at massive scale. It is observed that numerous social networks including Youtube (over 190 million users), Orkut (over 62 million), LiveJournal (over 5.5 million) [1], Cyworld (over 12 million), Myspace (over 130 million) [2], and Flickr (over 1 million) [3, 4] exhibits two interesting and important features: 1) a small mean degrees of separation (so called the small-world phenomenon) and 2) power-law scaling in node relationship, i.e., node degree distribution.

However, so far the research on networks with two characteristics aforementioned mainly focuses on measurement-based analysis of structural and topological characteristics, yet little is known about their impact on network performance metrics such as throughput, delay, etc. As one exception, Cha et al. [4] investigate the information propagation process in Flickr and show that social structure and behaviors may incur subtle latency, but the explicit mechanism is still not clear.

This paper presents a first look into the throughput capacity of large scale wireless networks with heterogeneous node distribution and relationship from a theoretical perspective. Previous works related to this [5, 6, 7, 8, 9, 10, 11, 12, 13, 14] have already pointed out that the heterogeneity brings new challenges as well as opportunities to system design, and new communication protocols may be conceived to exploit the underlying network structure for better performance. However, the feasible performance gains in terms of throughput capacity as well as the optimal communication schemes have not been investigated so far. In this paper, we bridge the theoretical analysis of fundamental scaling laws of wireless networks consisting an increasing number of nodes with the insights already gained through practical protocol development. By doing so, we provide a theoretical foundation to the design of intelligent scheduling and routing schemes which exploit the geographical locations of nodes as well as node relationship, analytically showing the potential of such schemes in terms of throughput capacity.

In particular, to reproduce the aforementioned two major features of such large scale networks, we deploy a simple but novel model, where the node relationships are constructed through network geography, i.e., the spatial properties of nodes. The network is assumed to be comprised by \( n + 1 \) uniformly distributed wireless nodes, and the probability of befriending a particular node is inversely proportional to the \( \alpha \)th power of the number of closer nodes. Liben-Nowell et al. [15] show that these rank-based models hold for real world wireless networks and leads to a navigable small world. For each node a random number of friends, which follows
power-law distribution with parameter $\beta$, will be selected independently.

We study two kinds of common traffic patterns, i.e., unicast and multicast. The first type represents messaging service between two friends while in the latter pattern information is broadcasted to all friends of the source, such as tweets in Twitter and posts in Facebook. We show that the per-node unicast capacity is $\Theta(1/\sqrt{n})$\footnote{The order notation $\Theta(\cdot)$ hides polylogarithmic factors for better readability. Refining results are available in Section III and IV.} when $\alpha \leq 1$, $\Theta(n^{\alpha - 3/2})$ when $1 < \alpha < 3/2$, and $\Theta(1)$ when $\alpha > 3/2$. When $\alpha = 0$, the per-node multicast capacity is $\Theta(1/n)$ when $\beta \leq 1$, $\Theta(n^{\beta - 2})$ when $1 < \beta < 3/2$, and $\Theta(1/\sqrt{n})$ when $\beta > 3/2$. In the more general but intricate case that $\alpha$ is arbitrary, we conjecture the multicast capacity to be $\Theta(n^{\alpha + \beta - 3})$ when $\alpha, \beta \in [1, 3/2]$. The results above are significantly better than the capacity of networks with classic uniform traffic, which is $\Theta(1/n)\log n$ in the unicast case and $\Theta(n^{\beta - 5/2})$ in the multiple unicast case, thanks to the traffic locality and multicast gains resulted from the underlying network structure. The corresponding capacity-achieving communication schemes are also discussed.

It is worth noting that both the rank-based model and the power law node degrees are heavy-tailed distributions. Heavy-tailed distributions are powerful modeling tools in realistic settings, but are often difficult for analysis because they imply a great degree of variations in the system, i.e., some of the source-destination pairs are close neighbors while some may be very far away. As well, some of the nodes have extremely many followers (such as celebrities) while some other may only have a few. However, our results show that despite the great heterogeneities in the network, a uniform optimal performance can be guaranteed.

1.1 Related Works

The asymptotic capacity of wireless networks is first studied in [16], where Gupta and Kumar show that the maximal unicast throughput achievable by each node for a uniformly distributed destination is $\Theta(1/\sqrt{n \log n})$. Grossglauser and Tse [17] later introduce mobility to the nodes and show that by employing a store-carry-forward paradigm, capacity can be improved to $\Theta(1)$, at the expense of increased delay. A series of works [18, 19, 20] have then been focusing on the analysis of optimal throughput-delay tradeoff under different mobility models.

Among numerous papers following Gupta and Kumar’s framework to investigate the capacity of various kinds of specific wireless networks, the most related ones consider heterogeneous networks. Garetto et al. [21, 22] study the capacity scaling in ad hoc networks with heterogeneous nodes mobility. Alfano et al. [23, 24] consider the case that nodes are distributed heterogeneously according to a shot-noise Cox process, such that clusters may be formed. Huang and Wang [25] analyze a network consisting of nodes with heterogeneous priority. However, to the best of the authors’ knowledge, this is the first work that takes into account the heterogeneity in both node distribution and relationship and studies their impact on throughput capacity.

Finally, multicast in traditional wireless ad hoc network is investigated by Li [26], who shows that per-node multicast capacity is $\Theta(1/\sqrt{nk \log n})$ when $k = O(n/\log n)$ and $\Theta(1/k \sqrt{\log n})$ when $k = \omega(n/\log n)$, where $k$ is the number of destinations per multicast session. Wang et al. [27] generalize the result to anycast traffic pattern and Mao et al. [28] study multicast networks with infrastructure support.

The rest of the paper is organized as follows. We introduce the system model in Section 2, and derive the unicast capacity in Section 3. Section 4 discusses multicast and we conclude the paper in Section 5.

2. SYSTEM MODEL

Throughout this paper we denote the probability of an event $E$ as $\Pr(E)$ and say $E$ happens with high probability (w.h.p.) if $\lim_{n \to \infty} \Pr(E) = 1$. By convention, we use $\{c_i\}$ to denote some positive constants independent of $n$.

2.1 Network Topology

We define the network extension $\mathcal{O}$ to be a unit torus. The size normalization and wrap-around conditions are common technical assumptions adopted in previous works to avoid tedious technicalities. These assumptions will not change the main results of this paper. $n + 1$ nodes with wireless communication capability spread in the network and exchange information in an ad hoc manner. Their locations are $\{X_i\}_{i=1}^n$, which are a series of independent random variables uniformly distributed in $\mathcal{O}$. At times nodes may be denoted by their positions, i.e., we refer to node $i$ as $X_i$.

2.2 Communication Model

We assume all nodes share a wireless channel with bandwidth $W$ bps. We base our analysis on the following classic wireless interference model which governs direct radio transmissions between nodes.

Definition 1. Protocol Model [16]: all nodes use a common transmission range $R_T$ for all their wireless communication. A wireless transmission from node $i$ to $j$ is successful only if: 1) $\|Z_i(t) - Z_j(t)\| \leq R_T$; and 2) For every other node $k$ that is simultaneously transmitting, follows, $\|Z_k(t) - Z_j(t)\| \geq (1 + \Delta)R_T$, where constant $\Delta$ defines the area of guard zone.
2.3 Node Relationship and Traffic Pattern

It is worth noting that modeling node relationship and network topology is rather challenging arguably due to the complicated nature of nodes’ behaviors, the diverse structure characteristics observed from real data sets and the massive scale. Sala et al. [29] show that among the existing models for synthetic graphs, only very few of them may capture the full structural characteristics or produce results with high fidelity. Furthermore, most of these models are based on numerical methods and will incur high computational and memory complexity.

We approach the modeling of node relationship in a totally new and novel way that is motivated by a geographical perspective. Both everyday life experience and real data traces from online social networks [15, 4, 30] indicate that friendships, communication patterns are closely related to geography and are usually highly localized. We adopt the following rank-based model [15] to characterize the relation between friendship and node location. Consider two nodes $i$ and $j$, define the rank of $j$ with respect to $i$ as:

$$\text{Rank}_i(j) = |\{k : D(k, i) < D(i, j)\}|$$

where $D(\cdot, \cdot)$ is the distance between two nodes. Then we model the probability that $j$ is a friend of $i$ as

$$\Pr(i \rightarrow j) \propto \frac{1}{\text{Rank}_i(j)}$$

where $\alpha \geq 0$. Denoting for short $G_1 = \sum_{j=1}^{n+1} 1/j^\alpha$, the distribution law is

$$\Pr(i \rightarrow j) = \frac{1}{G_1 \text{Rank}_i(j)} \quad (1)$$

Liben-Nowell et al. [15] show that this model accounts for the majorities of the friendships in the LiveJournal online community. Further theory [31] suggests that the model indeed guarantees small-world properties, such that with geographical information only, a friendship chain with at most $\Theta(\log^2 n)$ hops can be established between an arbitrary source node and a target person chosen uniformly at random from the whole population. This clear-cut property remarkably coincides with the fact that shortcuts can be found between two arbitrary nodes in LiveJournal with only geographical information.

Another important feature is the power-law degree distribution [1, 2, 3, 4]. We assume $K_i$, the number of friends of a particular node $i$, is drawn according to Zipf distribution.

$$\Pr(K_i = k) = \frac{1}{G_2 k^\beta}$$

where $G_2 = \sum_{j=1}^{n+1} 1/j^\beta$ is the normalizing factor and $\beta \geq 0$ is the power-law parameters. $K_i$ friends are chosen independently according to the rank-based model (1). We focus on the case that $\beta > \alpha$ such that no node will not be repetitively chosen in $K_i$ trials w.h.p.\(^2\)

We study two kinds of major traffic patterns in such networks, i.e., unicast and multicast. The first type of traffic represents (private) messaging service between two friends, i.e., sources will select their destinations with the rank-based model (1). In the latter traffic pattern information is broadcasted to all nodes that have relationship with the source, such as tweets in Twitter and posts in Facebook. Therefore multiple friends will be chosen according to the above power-law model.

We note that our model is not a perfect characterization of general graphs. For example, some of the friendships in LiveJournal appears to be geography independent and may be better explained in other dimensions such as occupations, age, etc. However, a complete reproduction of all features in a realistic network is too difficult, if at all possible, and we believe it is beneficial to make the proper simplifications towards a tractable model and a meaningful look into the throughput capacity in networks with heterogeneity.

2.4 Capacity Definition

Definition 2. Feasible multicast throughput: Per-node throughput $g(n)$ is said to be feasible if there is a spatial and temporal scheme for scheduling transmissions, such that by operating the primary network in a multi-hop fashion and buffering at intermediate nodes when awaiting transmission opportunities, every source can send $g(n)$ bits/sec to its $K_i$ chosen destination nodes. That is, there is a $T < \infty$ such that in every time interval $[(i-1) \cdot T, i \cdot T]$, every source can sent $T \cdot g(n)$ bits to each of its $K_i$ destinations.

Definition 3. Asymptotic per-node multicast capacity $\lambda_\alpha(n)$ of the network is said to be of order $\Theta(g(n))$ if there exist two positive constants $c_1$ and $c_2$ such that:

$$\lim_{n \to \infty} \Pr\{\lambda_\alpha(n) = c_1 g(n)\ \text{is feasible}\} = 1$$

$$\lim_{n \to \infty} \Pr\{\lambda_\alpha(n) = c_2 g(n)\ \text{is feasible}\} < 1$$

Similarly we define the asymptotic per-node unicast capacity $\lambda_\alpha(n)$.

2.5 Notations

In table 1, we list all the parameters that will be used in later analysis, proofs and discussions.

3. MAIN RESULTS

A graphical representation of our results is reported in Figures 1 and 2, respectively. We adopt the order notation $\tilde{\Theta}(\cdot)$ to hide poly logarithmic factors for better readability. Refined results are available in Section V.
<table>
<thead>
<tr>
<th>Notation</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n + 1$</td>
<td>The total number of nodes in the network.</td>
</tr>
<tr>
<td>$W$</td>
<td>The total transmission bandwidth available.</td>
</tr>
<tr>
<td>$R_T$</td>
<td>transmission range</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>Power law parameter indicating the strength of the relation between two nodes.</td>
</tr>
<tr>
<td>$\beta$</td>
<td>Power law parameter indicating the degree of a node.</td>
</tr>
<tr>
<td>$\lambda_u(n)$</td>
<td>Asmptotic per-node unicast capacity.</td>
</tr>
<tr>
<td>$\lambda_m(n)$</td>
<td>Asmptotic per-node multicast capacity.</td>
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</tbody>
</table>

Figures 1 plots the per-node unicast capacity $\lambda_u(n)$ achieved versus different values of parameter $\alpha$. As a counterpart, Figure 2 represents the per-node multicast capacity $\lambda_m(n)$, which exhibits different scaling behavior with different parameters $\beta$. A common phenomenon from both the figures is that the capacity decreases as $\alpha$ and $\beta$ increases. And the maximum capacity can be achieved in the range $0 < \alpha < 1$ for unicast and $\beta > 3/2$ for multicast. The reason behind is that the transmission length in both cases is greatly reduced, a characteristic in networks that people who have a stronger relationship tend to locate more closer to each other.

![Figure 1: The per-node unicast capacity $\lambda_u(n)$ versus $\alpha$.](image1.png)

![Figure 2: The per-node multicast capacity $\lambda_m(n)$ versus $\beta$.](image2.png)

4. UNICAST

4.1 Traffic Locality

Comparing with classic unicast networks, the traffic pattern in our model is significantly different because the destinations are selected according to the rank-based model, which will result in a certain degree of traffic locality. Intuitively, as parameter $\alpha$ increases, sources will be more likely to befriend a node located in closer proximity, and therefore less distance or hops are needed to be covered in the packet delivery process. This amounts to a smaller interference per traffic flow, and in terms imply a larger degree of transmission concurrency can be achieved. As a result, the unicast capacity is increased.

However, the non-uniformity of the traffic pattern will cause significant difficulty in analysis. In order to proceed, we first need to establish some important properties and implications of the ranked-based model. Denote $Y_i$ as the destination selected by $X_i$, the following lemma shows the distribution of the distance between $Y_i$ and $X_i$ conditioning on rank.

**Lemma 1.** Consider a generic node $i$, conditioning on the event that $\text{Rank}_i(Y_i) = r$, the probability density function (PDF) of random variable $D(X_i,Y_i)$ is:

$$f_{D|m}^{(\text{D})}(d) = \frac{n!}{(r-1)!(n-r)!}2\pi^r d^{r-1}(1 - \pi d^2)^{n-r-1}$$ (2)

for $0 \leq d \leq \frac{1}{\sqrt{\pi}}$.

**Proof.** Let $X_j, j \neq i$ be an arbitrary node in the system. According to our model, $X_j$ is uniformly distributed, and the cumulative distribution function (CDF)
of \(D(X_i, X_j)\) follows:\(^3\)

\[
F^{(D)}(d) = \Pr\{D(i,j) \leq d\} = \int_{x^2+y^2 \leq d^2} 1 \, dx dy = \pi d^2, \quad 0 \leq d \leq \frac{1}{\sqrt{\pi}}
\]

And the corresponding PDF is:

\[
f^{(D)}(x) = 2\pi d, \quad 0 \leq d \leq \frac{1}{\sqrt{\pi}}
\]

Now consider the mechanism of the rank-based model, conditioning on the \(r\)th order statistics \([32]\) of \(n\) independent and identically distributed (i.i.d.) \(\{D(i,j)\}_{j \neq i}\). By convention we denote \(D(X_i, X_j)\) as \(D_{r:n}\).

The CDF of \(D_{r:n}\) may be obtained by standard technique,

\[
F^{(D)}_{r:n}(d) = \Pr\{D_{r:n} \leq d\} = \Pr\{\text{at least } r \text{ of } D(i,j)\'s \text{ are at most } d\} = \sum_{k=r}^{n} \Pr\{\text{exactly } k \text{ of } D(i,j)\'s \text{ are at most } d\} = \sum_{k=r}^{n} \binom{n}{k} (F^{(D)}(d))^k (1 - F^{(D)}(d))^{n-k}
\]

\[
= \sum_{k=r}^{n} \binom{n}{k} (\pi d)^k (1 - \pi d)^{n-k}
\]

\[
= \int_{0}^{\pi d^2} \frac{n!}{(r-1)!(n-r)!} t^{r-1}(1-t)^{n-r} \, dt = \int_{0}^{d} \frac{n!}{(r-1)!(n-r)!} 2\pi^r t^{2r-1}(1 - \pi t^2)^{n-r} \, dt
\]

Where equation (3) holds due to the fact that,

\[
\sum_{k=r}^{n} \binom{n}{k} t^{r}(1-p)^{n-r} = \int_{0}^{p} \frac{n!}{(r-1)!(n-r)!} t^{r-1}(1-t)^{n-r} \, dt, \quad 0 < p < 1
\]

which may be proved by repeated integration by parts.

From (4) we observe the density function of \(D_{r:n}\) is

\[
f^{(D)}_{r:n}(d) = \frac{n!}{(r-1)!(n-r)!} 2\pi^r d^{2r-1}(1 - \pi d^2)^{n-r}
\]

\(^3\)More rigorously, taking the four corners of the square extension into account, the CDF should be:

\[
\Pr\{D \leq d\} = \begin{cases} \pi d^2, & 0 \leq d \leq \frac{1}{2} \\ \pi d^2 \left( \frac{n-4}{n} \arccos \left( \frac{d}{2} \sqrt{n} \right) + \sqrt{4d^2 - 1} \right), & \frac{1}{2} < d \leq \frac{\sqrt{n}}{2} \end{cases}
\]

This piecewise function is awkward for presentation and we therefore simplify the extension to be a disk in the lemma derivation. However, it can be easily shown that this slight modification does not have any significance in order sense.

for \(0 \leq d \leq \frac{1}{\sqrt{\pi}}\). \(\square\)

In the next step we characterize the conditional expectation of \(D(X_i, Y_i)\).

**Lemma 2.** Conditioning on the event that \(\text{Rank}_i(Y_i) = r\) and denote \(E\) as expectation, then

\[
E\{D(X_i, Y_i)|r : n\} = \sqrt{\pi} \frac{\Gamma(n+1)}{\Gamma(n+3/2)} \frac{\Gamma(r+1/2)}{\Gamma(r)} \sim \sqrt{r/n}
\]

**Proof.** By definition,

\[
E\{D_{r:n}\} = \int_{0}^{\pi d^2} x f^{(D)}_{r:n}(d) \, dx
\]

\[
= \int_{0}^{1} \frac{n!}{(r-1)!(n-r)!} 2\pi^r x^{2r}(1 - \pi x^2)^{n-r} \, dx
\]

\[
= \int_{0}^{1} \frac{n!}{(r-1)!(n-r)!} \frac{2}{\sqrt{\pi}} t^{2r}(1 - t^2)^{n-r} \, dt = \frac{B(r+1/2, n-r+1)}{\sqrt{\pi} B(r, n-r+1)} = \frac{1}{\sqrt{\pi}} \frac{\Gamma(n+1)}{\Gamma(n+3/2)} \frac{\Gamma(r+1/2)}{\Gamma(r)}
\]

where \(B(\cdot, \cdot)\) is the complete beta function:

\[
B(p, q) = \int_{0}^{1} t^{p-1}(1-t)^{q-1} \, dt, \quad p, q > 0
\]

and \(\Gamma(\cdot)\) is the complete gamma function:

\[
\Gamma(p) = \int_{0}^{\infty} e^{-t} t^{p-1} \, dt, \quad p > 0
\]

With Stirling’s approximation of gamma function \([33]\),

\[
\Gamma(p) = \sqrt{2\pi p} \left( \frac{p}{e} \right)^p \left( 1 + O\left( \frac{1}{p} \right) \right)
\]

holds,

\[
E\{D_{r:n}\} \sim \frac{\sqrt{n+1} \Gamma(n+1)}{\sqrt{n+\frac{1}{2} \Gamma(n+\frac{1}{2})}} \frac{\sqrt{r+\frac{1}{2} \Gamma(r+\frac{1}{2})}}{\sqrt{\pi} \Gamma(r)}
\]

\[
\sim \left( \frac{n+1}{n+\frac{1}{2}} \right)^{n+1} \frac{1}{(n+\frac{3}{2})^2} \frac{r+\frac{1}{2}}{r} \left( r+\frac{1}{2} \right)^{\frac{1}{2}}
\]

\[
\sim \left( \frac{1}{2(n+\frac{1}{2})} \right)^{n+1} \left( 1 + \frac{1}{2r} \right) \sqrt{\frac{r}{n}} \sim \frac{1}{\sqrt{r/n}}
\]

\(\square\)

Before proceeding we introduce a useful lemma on estimating the partial sum of \(p\)-series by the integral test inequality.

\[\text{\LARGE 5}\]
4.2 Upper Bound of Capacity

Based on the spatial characteristics of the traffic pattern, an upper bound of capacity can be derived by establishing the relation between throughput and the average distance that the packets need to be relayed.

**Lemma 3.** Suppose $g(x)$ is a continuous decreasing function and $g(x) > 0$ for all $x \geq 1$, then

$$\int_1^n g(x)dx \leq \sum_{m=1}^{n-1} g(m) \leq g(1) + \int_1^{n-1} g(x)dx$$

We conclude this subsection with the expectation of $D(X_i, Y_i)$ in general.

$$\mathbb{E}\{D(X_i, Y_i)\} = \mathbb{E}_r\{\mathbb{E}\{D(X_i, Y_i)|r : n\}\}$$

Recall that $G_1$ is the normalizing factor in (1). Setting $g(x) = 1/x^\alpha$, by Lemma 3 it is clear that,

$$G_1 = \begin{cases} 
\Theta(1) & \alpha > 1 \\
\Theta(\log n) & \alpha = 1 \\
\Theta(n^{1-\alpha}) & 0 < \alpha < 1 
\end{cases}$$

Similarly, by setting $g(r) = \sqrt{n}/r^\alpha$,

$$\sum_{r=1}^{n} \frac{1}{r^\alpha} = \begin{cases} 
\Theta(1) & \alpha > 3/2 \\
\Theta(\log n) & \alpha = 3/2 \\
\Theta(n^{3/2-\alpha}) & 0 < \alpha < 3/2 
\end{cases}$$

Combing the above formulas we have

$$\mathbb{E}\{D(X_i, Y_i)\} \sim \begin{cases} 
\frac{1}{\sqrt{n}} \log n/\sqrt{n} & \alpha > 3/2 \\
n^{1-\alpha} & 1 < \alpha < 3/2 \\
1/\log n & \alpha = 1 \\
1 & 0 \leq \alpha < 1 
\end{cases}$$

(5)

**4.2 Upper Bound of Capacity**

Consider that at time $t$ generic nodes $i$, $j$ are transmitting directly to nodes $k$ and $l$, respectively. According to the protocol interference model, the following conditions must hold in order for successful reception.

$$d(X_j, X_k) \leq (1 + \Delta)d(X_i, X_k)$$

$$d(X_i, X_l) \leq (1 + \Delta)d(X_j, X_l)$$

Therefore,

$$d(X_j, X_l) \geq d(X_j, X_k) - d(X_i, X_k) \geq \Delta d(X_i, X_k)$$

Likewise, we have

$$d(X_i, X_j) \geq \Delta d(X_i, X_l)$$

Hence,

$$d(X_i, X_j) \geq \frac{\Delta}{2} (d(X_i, X_k) + d(X_j, X_l))$$

The above inequality states that as a consequence of the protocol model, disks of radius $\Delta/2$ times the transmission range centered at the transmitter are disjoint from each other. This "transmission consumes area" argument serves as one corner stone of the upper bounds on achievable throughput. Notice that 1) $\mathcal{O}$ has unit area; 2) for each of these disjoint disks, at least $1/4$ of it must lie within $\mathcal{O}$ and 3) transmitting a packet requires $1/eP$ duration of time, therefore,

$$\frac{1}{4} \sum_{p=1}^{N_p} \sum_{h=1}^{h_p} \pi \left[ \frac{\Delta h_p}{2} \right]^2 \leq c_P WT$$

(7)

By Cauchy-Schwarz Inequality,

$$\left[ \sum_{p=1}^{N_p} \sum_{h=1}^{h_p} h_p \right]^2 \leq \left[ \sum_{p=1}^{N_p} \sum_{h=1}^{h_p} h_p^2 \right] \left[ \sum_{p=1}^{N_p} \sum_{h=1}^{h_p} 1 \right]$$

(8)

where observing the fact that at any time there are at most $n+1$ transmissions in the network, the last factor can be reduced to,

$$\frac{N_p}{p=1} \sum_{h=1}^{h_p} 1 = \frac{N_p}{p=1} h_p \leq c_P WT(n + 1)$$

(9)

Substituting (6)-(9) we have,

$$\frac{16c_P WT}{\pi \Delta^2} \geq \sum_{p=1}^{N_p} \sum_{h=1}^{h_p} (h_p^2)$$

$$\geq \left[ \frac{\sum_{p=1}^{N_p} \sum_{h=1}^{h_p} h_p}{\sum_{h=1}^{h_p} h_p} \right]^2 \frac{c_P \lambda(x(n+1))T\mathcal{D}}{c_P WT(n + 1)}$$

It is also possible to establish similar observations under the generalized physical model [34].
3. Denote a straight line segment connecting the source to nodes inside the same cell or neighboring cells.

Lemma 5. Let \( \alpha(n) = K \log n/n, \) for any \( K > 1, \)
uniformly over \( \mathcal{O} \) it holds that each cell contains at least
one nodes but no more than \( K e \log n \) nodes w.h.p.

Lemma 5 ensures connectivity and as a result, we choose \( \alpha(n) = \Theta(\log n/n) \) for minimal interference. The next
lemma is a simple consequence of the protocol model and the well-known fact about vertex coloring of graphs of bounded degree, see [16] for an example of proof.

**Lemma 6.** With the TDMA scheme described above, each cell has a constant fraction of time to be active
(See illustration 4.).

![Figure 3: Multi-hop scheme from \( X_i \) to \( Y_i \).](image)

To 1) choose a proper \( a(n) \) which guarantees network connectivity; 2) show that the TDMA scheme allows high spatial reuse and concurrent transmissions and 3) analyze the service load of the cells. We begin with a well-known lemma [34] which is a standard application of Chernoff bounds [35].

Thus,

\[
\lambda \leq \frac{4W}{\Delta \mathcal{D}} \frac{1}{\sqrt{\pi(n+1)}} \sim \frac{1}{\mathcal{D} \sqrt{n}}
\]

With Lemma 4 and (5), we have

**Theorem 1.** Under the ranked-based model, an upper bound of per-node unicast capacity is

\[
\lambda_u(n) \sim \begin{cases} 
O(1) & \alpha > 3/2 \\
O(1/\log n) & \alpha = 3/2 \\
O(n^{\alpha-3/2}) & 1 < \alpha < 3/2 \\
O(\log n/\sqrt{n}) & \alpha = 1 \\
O(1/\sqrt{n}) & 0 \leq \alpha < 1
\end{cases}
\]

4.3 Capacity Achieving Scheme

In this subsection we will show that a straightforward cell tessellation multi-hop relaying scheme suffice to achieve the capacity upper bound in Theorem 1.

**Optimal Scheme for Unicast:**

1. Tessellate \( \mathcal{O} \) into squarelets (cells) with area \( a(n) \).
2. Employ a cellular time-division multi-access (TDMA) transmission scheme such that each cell is scheduled to be active regularly according to cell time-slots. When a cell is activated, nodes within it are allowed to transmit to nodes inside the same cell or neighboring cells.
3. Denote a straight line segment connecting the source \( X_i \) and the destination \( Y_i \) as S-D line. Sources send their packets to destinations hop by hop along the cells which the S-D lines intersect.
4. When a cell is scheduled to be active, it transmit a single packet for each of the passing through S-D lines. This is again accomplished by adopting a TDMA scheme such that the cell time-slot is further divided into sub packet time-slots.

The simple scheme above is similar to the one used in [18]. To establish the optimality of the scheme, we need

![Figure 4: 9-TDMA scheme where the whole network is divided into clusters with equal area. Each 9 groups are categorized as a group. All the grey cells in each group (numbered with 1) can transmit simultaneously in a time slot. In the next time slot all the cells numbered with 2 transmit and so on.](image)
\[\alpha(\sqrt{a(n)}). \text{ Define } H = \sum_{i=1}^{n+1} h_i, \text{ i.e., the sum of hops required for each source in the network to send a single packet to its destination.} \]

Now consider a generic cell \( V \) and define the Bernoulli random variable \( I_i \) for S-D pairs \( 1 \leq i \leq n+1 \) and \( 1 \leq k \leq h_i \), such that \( I_i = 1 \) if the hop originated from S-D pair \( i \) intersects cell \( V \). Therefore the total number of S-D lines passing through the cell is 
\[
I = \sum_{i=1}^{n+1} \sum_{h=1}^{h_i} I_i.
\]
Notice that all \( I_i \) are identical distributed; further, \( I_i \) and \( I_j \) are pairwise independent if \( i \neq j \). However, \( I_i \) and \( I_i \) are dependent since a S-D line can intersect a given cell for at most one time.

\( I \) is the major quantity of interest because it characterizes the relaying load of a cell. In the following we shall first determine its expectation, and then bound the tail probability.

\[
\mathbb{E}[I] = \mathbb{E}_H[\mathbb{E}[I|H]]
\]
\[
= \mathbb{E}_H\left[ \sum_{i=1}^{n+1} \sum_{h=1}^{h_i} \mathbb{E}[I_i^h] \right]
\]
\[
= \mathbb{E}_H[H\mathbb{E}[I_i^h]] = \mathbb{E}_H[H\mathbb{a}(n)]
\]
\[
= a(n) \mathbb{E}\left[ \sum_{i=1}^{n+1} \frac{d_i}{\sqrt{a(n)} + 1} \right]
\]
\[
= (n + 1)(\mathbb{E}[d_i]\sqrt{a(n)} + a(n))
\]
where in the third equation, \( \mathbb{E}[I_i^h] \) equals \( a(n) \) since by the symmetry of the torus, any hop is equally likely to originate from any of the \( 1/a(n) \) cells.

The remaining part of the proof essentially needs to show that \( I \) will not deviate too much from \( \mathbb{E}[I] \) w.h.p.
A common technique is to apply Chernoff bounds to obtain the tail probability of \( I \), which is a sum of random variables (RVs). However, two major challenges are: 1) \( I \) is a sum of random number of RVs and 2) these RVs are not independent. Traditional Chernoff bounds fail under these conditions, whereas we are going to prove that a similar bound still holds for our probability structure.

First it is helpful to bound the range of \( H \). We claim that w.h.p. \((1 - \epsilon)\mathbb{E}[H] < H < (1 + \epsilon)\mathbb{E}[H] \), for any constant \( \epsilon > 0 \). Indeed, define by short \( P(H, \epsilon) = \Pr\{ |H - \mu_H| < \epsilon\mu_H \} \), with Chebyshev’s inequality,

\[
P(H, \epsilon) \geq 1 - \frac{\text{Var}(h_i)}{(n + 1)\epsilon^2\mathbb{E}^2[h_i]}
\]
where \( \text{Var}(h_i) \) is the variance of \( h_i \) (for all \( i \)). And,

\[
\text{Var}(h_i) = \mathbb{E}[h_i^2] - \mathbb{E}^2[h_i]
\]
\[
= (\mathbb{E}[d_i^2] - \mathbb{E}^2[d_i])/a(n)
\]
Since \( d_i < 2/\sqrt{2} \), \( \text{Var}(h_i) = O(1/a(n)) = O(n/\log n) \).

Therefore, for some constant \( c \), it holds

\[
P(H, \epsilon) \geq 1 - \frac{1}{c^2\mathbb{E}^2[h_i]\log n} \to 1, \quad \text{as } n \to \infty
\]

Let \( \tilde{H} \) be the event that \( H \) is bounded by \((1 \pm \epsilon)\mu_H \).

Now we construct random variable \( \tilde{l} = \sum_{i=1}^{n+1} \tilde{I}_i \), where \( \tilde{I}_i \) are i.i.d. Bernoulli random variables with the same distribution as \( I_i \). Because of the dependency between \( I_i \) and \( \tilde{I}_i \) (i.e., the event that both \( I_i \) and \( \tilde{I}_i \) equals 1 is not possible), \( \tilde{l} \) is stochastically larger than \( I \). By the property of stochastic ordering, for any increasing function \( \phi(\cdot) \) such that the expectation\(^5 \) exists, we have

\[
\mathbb{E}[\phi(I)] \leq \mathbb{E}[\phi(\tilde{l})]
\]

Let \( t \) be an arbitrary positive constant, define for short \( P^+(I, \delta) = \Pr\{ I \geq (1 + \delta)\mathbb{E}[I] \} \), proceed with the main steps in the proof of Chernoff bounds:

\[
P^+(I, \delta) = \Pr\{ I \geq (1 + \delta)\mathbb{E}[I] \}
\]
\[
= \Pr\{ \exp(tI) \geq \exp((1 + \delta)\mathbb{E}[I]) \}
\]
\[
\leq \frac{\mathbb{E}\{ \exp(tI) \}}{\exp((1 + \delta)\mathbb{E}[I])}
\]
\[
\leq \frac{\mathbb{E}\{ \exp((1 + \delta)\mathbb{E}[I]) \}}{\exp(t(1 + \delta)\mathbb{E}[I])}
\]

where (11) is the consequence of Markov inequality and (12) holds from (10). Exploiting the independence between \( I_i \), yields

\[
\mathbb{E}\{ \exp(tI) \} = \mathbb{E}\left[ \prod_{i=1}^{n+1} \exp(tI_i) \right] = \mathbb{E}_H\left[ \prod_{i=1}^{n+1} \mathbb{E}[\exp(tI_i)|H] \right]
\]

Let \( p = a(n) = \Pr\{ I_i = 1 \} \) be the success probability,

\[
P^+(I, \delta) < \frac{\mathbb{E}_H\{ \prod_{i=1}^{n+1} [pe^t + 1 - p] \} \exp(t(1 + \delta)\mathbb{E}[I])} {\exp(t(1 + \delta)\mathbb{E}[I])}
\]
\[
= \frac{\mathbb{E}_H\{ \prod_{i=1}^{n+1} \exp(\epsilon^t - 1) \} \exp(t(1 + \delta)\mathbb{E}[I])} {\exp(t(1 + \delta)\mathbb{E}[I])}
\]
\[
= \frac{\exp\{\sum_{i=1}^{n+1} \mathbb{E}[I] \} \exp(\epsilon^t - 1) \exp(t(1 + \delta)\mathbb{E}[I])} {\exp(t(1 + \delta)\mathbb{E}[I])} \Pr\{ \tilde{H} \} + \Pr\{ \tilde{H}^c \}
\]
\[
= \frac{\exp((\epsilon^t - 1)(1 + \epsilon)\mathbb{E}[I])}{\exp(t(1 + \delta)\mathbb{E}[I])} \Pr\{ \tilde{H} \} + \Pr\{ \tilde{H}^c \}
\]

(15)

where (13) holds due to inequality \( 1 + x < e^x \); (14) holds due to the law of total probability and the monotonicity of \( \exp \) and; (15) holds because of the fact that \( \mathbb{E}[I] = \mathbb{E}[H] \mathbb{E}[\tilde{I}] \). Lastly, by setting \( t = \log(1 + \delta) \) and noticing

\(^5\)It is easy to show that the same result holds for conditional expectation in our case. Refer to [36] for more about stochastic ordering.
that $\mathbb{E}[I] = \mathbb{E}[\tilde{I}]$, we recover a bound which is arbitrarily closed to the classic Chernoff bound,

$$P^+(I, \delta) < \left[ \frac{e^{(1+\delta)^2}}{(1+\delta)^{(1+\delta)}} \right]^{\mathbb{E}[I]} \Pr\{\mathcal{H}\} + \Pr\{\mathcal{H}^c\}$$

Now we may choose $\delta$ as $^6$,

$$\delta = \left( \frac{4 \log n^2}{\mathbb{E}[I]} \right)^{\frac{1}{2} - \epsilon} < \left( \frac{8 log n}{na(n)} \right)^{\frac{1}{2} - \epsilon} = O(1)$$

Thus,

$$P^+(I, \delta) = \Pr\{I \geq (1 + O(1))\mathbb{E}[I]\}$$

$$< \frac{1}{n^2} \Pr\{\mathcal{H}\} + \Pr\{\mathcal{H}^c\}$$

Therefore, for a general cell $V$, the number of S-D lines passing through it is upper bounded by $\Theta(\mathbb{E}[I]) = O\left(n \mathbb{E}[D(X_i, Y_j)] \sqrt{a(n)} + na(n)\right)$ with probability $1 - 1/n^3$ conditioning on event $\mathcal{H}$. Since $\mathcal{H}$ happens w.h.p. and with the uniform bound the above bound holds uniformly for all cells in $O$ with probability $1 - 1/n$. This completes the proof. \hfill \Box

**Remark 1.** Lemma 7 has an interesting implication: notice that the rank-based model follows power law and is in fact a heavy-tailed distribution. Therefore the tail of $D(X_i, Y_j)$ plays an important role and cannot be ignored. Hence the number of hops $h_i$ is likely to deviate much from its expectation and its variance might not be bounded. These observations lead to concerns that whether the load of the cells are disproportional so bottlenecks may be formed in the network and capacity is reduced. The answer is no, according to Lemma 7, which shows that we still have nice convergence in probability uniformly over all cells in the network. This not only enables succinct capacity results to be obtained, but also implies that we can combat network heterogeneities and achieve load balancing using simple scheduling schemes.

Combining Lemma 5, 6 and 7, the following theorem is straightforward.

**Theorem 2.** The per-node throughput of the scheme for unicast is $\Omega\left( \frac{1}{n^{\epsilon}}(\mathbb{E}[D(X_i, Y_j)]\sqrt{a(n)} + a(n)) \right) = \Omega\left( (\log n + \sqrt{n \log n \mathbb{E}[D(X_i, Y_j)]})^{-1} \right)$. That is,

$$\lambda_n(n) \sim \begin{cases} 
\Omega(1/\log n) & \alpha > 3/2 \\
\Omega(1/\sqrt{n}) & \alpha = 3/2 \\
\Omega(n^{-\frac{2}{3}}) & 1 < \alpha < 3/2 \\
\Omega(1/\sqrt{n} \log n) & \alpha = 1 \\
\Omega(1/\sqrt{n} \log n^{4}) & \alpha < 1
\end{cases}$$

Comparing with the results in Theorem 1,

**Corollary 1.** The lower bounds in Theorem 2 is tight up to a logarithmic factor.

$^6$See [35] for details on the choosing technique.

**Remark 2.** In fact except for the case that $\alpha > 3/2$, the lower bounds in Theorem 2 differ from the upper bounds in Theorem 1 with only a factor of $1/\sqrt{\log n}$. This well-known difference is due to the simplicity of the cell tessellation scheme which employs a uniform transmission range of $\Theta(\sqrt{\log n})$. However, such slight performance drawback can be eliminated by adopting a more sophisticated tessellation scheme and applying percolation theory in routing [37]. Though it is not our main focus, we remark that it is not difficult to extend percolation theory based schemes to our framework and achieve a throughput that strictly meets the upper bounds.

## 5. MULTICAST

As a major kind of traffic in many online networks, the information from a source is often desired to be disseminated to all its corresponding friends, such as tweets in Twitter and posts in Facebook. In this section we discuss the network throughput capacity under such traffic pattern, i.e., multicast.

Multicast in traditional wireless network has been investigated in [26, 27]. Since relaying links may be shared by different destinations in a multicast session, multicast is more efficient and is able to achieve a better throughput than multiple unicast. A common approach for multicasting is to establish a spanning tree structure for routing.

However, comparing with traditional multicast, a major challenge that we face in studying multicast is that the number of destinations (friends) in each multicast session is assumed to be a random variable following power-law distribution, while in previous related works it is assumed to be a fixed quantity. Intuitively, this implies that the multicast tree in our case is more random in size.

The problem is further complicated by the rank-based destination selection mechanism. In previous works, destinations in a multicast session are assumed to be selected independently and uniformly from the population, whereas under our framework they are selected in a much more complicated way. In fact, the rank-based selection mechanism implies that the locations of the destinations are subtly dependent, which cause significant difficulties in the analysis of multicast trees. In order to proceed, we have to therefore limit our analysis to the special case that $\alpha = 0$, such that the destinations are selected independently and uniformly over the whole network. We note that this degenerated rank-based model is equivalent to the uniform model widely adopted in related works, and more importantly, the simplification enable us to focus on the impact of power-law distributed destination numbers without entangling with the delicate multicast tree generated by the rank-based model. Our conjecture on the more general case
of arbitrary $\alpha$ is proposed at the end of the section.

### 5.1 Upper Bound of Capacity

Consider a generic source $X_i$ and denote $Y_i^1, Y_i^2, \ldots, Y_i^K$, as its $K_i$ friends (destinations). Denote $\text{EMST}(U)$ as the Euclidean minimum spanning tree of set $U$, and $|\text{EMST}(U)|$ represents its total Euclidean edge lengths. The following lemma is a famous result on the asymptotic length of the Euclidean minimum spanning tree generated by i.i.d. point sets.

**Lemma 8.** Let $Y_i$, $1 \leq i < \infty$ be independent and identically distributed random variables in $\mathbb{R}^d$, $d \geq 2$, denote $M_k = \text{EMST}\{Y_1, \ldots, Y_k\}$, then with probability 1,

$$
\lim_{k \to \infty} M_k = c(d)n^{(d-1)/d} \int_{\mathbb{R}^d} f(x)^{(d-1)/d} dx
$$

where $f$ denotes the density of the distribution of $Y_i$ and $c(d) > 0$ is a constant independent of $k$.

With $d = 2$ and $f(x) = 1$ in our case, it is clear that $M_k \sim \sqrt{k}$ as $k \to \infty$. Then we may compute the average length of the Euclidean minimum spanning tree covering the source $X_i$ and its $K_i$ destinations, where $K_i$ follows power-law distribution with parameter $\beta$.

$$
\mathbb{E}[\text{EMST}(X_i, Y_1^1, Y_1^2, \ldots, Y_1^K)] = \mathbb{E}[K_i | \text{EMST}(X_i, Y_1^1, Y_1^2, \ldots, Y_1^K)| K_i = k]
$$

$$
\sim \frac{1}{G_2} \sum_{k=1}^{n} \frac{\sqrt{k}}{k^\beta}
$$

$$
\sim \begin{cases} 
1 & \beta > 3/2 \\
\log n & \beta = 3/2 \\
\sqrt{n}/\log n & 1 < \beta < 3/2 \\
\sqrt{n} & \beta = 1 \\
0 & 0 \leq \beta < 1 
\end{cases}
$$

Then with the minimum spanning tree we can establish a upper bound for the multicast capacity.

**Lemma 9.** Let $U_i = \{X_i, Y_i^1, Y_i^2, \ldots, Y_i^K\}$, if on average $|\text{EMST}(U)|$ is at least $\mathcal{D}$, then $\lambda_m(n) = O(1/\mathcal{D}\sqrt{n})$.

**Proof.** We define a multicast session as the duration from a packet arrives at the source till the packet is delivered to all destinations. Note that by the definition of Euclidean minimum spanning tree, in a multicast session the packet must be relayed over a distance of at least $|\text{EMST}(U)|$. Again consider a time interval $T$ which is large enough such that the total number of packets transmitted among all multicast sessions is $c_p \lambda_m(n+1)T$. Denote $h_p$ as the number of hops packet $p$ is relayed, $l_p^h$ as the transmission range of the $h$th hop, and $N_p = c_p \lambda_m(n+1)T$, it follows,

$$
\sum_{p=1}^{N_p} \sum_{h=1}^{h_p} l_p^h \geq c_p \lambda_m T \sum_{i=1}^{n+1} |\text{EMST}(U_i)|
$$

$$
\geq c_p \lambda_m T(n+1)\mathcal{D}
$$

(16) where (16) follows from the strong law of large numbers because $|\text{EMST}(U_i)|$ are i.i.d. distributed and $\mathbb{E}[|\text{EMST}|] < \infty$. The rest part of the proof is clear by applying the same logic as Lemma 4.

**Theorem 3.** If the number of destinations per multicast session follows power-law distribution with parameter $\beta$, an upper bound of the per-node multicast capacity is

$$
\lambda_m(n) \sim \begin{cases} 
O(1/\sqrt{n}) & \beta > 3/2 \\
O(1/\log n\sqrt{n}) & \beta = 3/2 \\
O(n^{\beta-2}) & 1 < \beta < 3/2 \\
O(\log n/n) & \beta = 1 \\
O(1/n) & 0 \leq \beta < 1 
\end{cases}
$$

### 5.2 Capacity Achieving Scheme

For multicast, the cell partition TDMA scheme that we employ in Section III.C is still highly efficient for scheduling active transmissions in the network. However, routing becomes a major issue in multicast since an optimal routing tree needs to be established. Our main idea is to first construct a Euclidean spanning tree using Prim’s algorithm, and then convert it to a multicast routing tree.

**Optimal Routing Tree for Multicast Session $U_i$:**

1. Construct a spanning tree using Prim’s algorithm:
   
   (1) Initially, nodes in $U_i$ form $K_i$ components. Set $g = 1$.
   
   (2) Partition the network into at most $K_i - g$ squares, such that their side length is $1/\sqrt{K_i - g}$.
   
   (3) Find a square that contains two nodes from two different connected components. Merge the two components by adding a edge between the two nodes.
   
   (4) Return the constructed tree $\text{ST}(U_i)$ if $g = K_i - 1$, otherwise $g := g + 1$ and goto step (2).

2. Tessellate the network extension into square cells with area $a(n)$. For each edge $uv$ in $\text{ST}(U_i)$, arbitrarily select a point $v$ in the cell that lies in the same row as $u$ and the same column as $v$, select a node in each of the cells that $uv$ and $vw$ crosses and connect these nodes to form a path from $u$ to $v$.

3. Combine the paths and remove cycles if any. Return the obtained multicast routing tree $\text{MRT}(U_i)$. Notice that in Step 1-(3), the square exists due to Pigeonhole principle, and in step 2, the node exists as long as the connectivity criterion $a(n) = \Omega(\log n/n)$ is satisfied.

Intuitively, we can use these cells (with area $a(n)$) as scheduling units and employ the TDMA scheduling scheme proposed in Section III.C, and route the packets along tree $\text{MRT}(U_i)$. In order to analyze throughput, it is important to study the “load” of each cell under these schemes.

**Lemma 10.** Given an arbitrary cell $s$, the probability that a multicast session $U_i$ is routed through $s$ is

$$
\frac{1}{\sqrt{K_i - g}}
$$

More strictly it should be $\left\lfloor \frac{1}{\sqrt{K_i - g}} \right\rfloor$, but we assume it to be an integer for the ease of presentation.
upper bounded by $c_3 |\text{EMST}(U)| \sqrt{a(n)}$.

**Proof.** Notice that the construction of MRT($U_i$) consists of $K_i - 1$ steps, and $s$ may be invoked in any of these steps. Denote $I_g$ as the indicator that whether $s$ is invoked in step $g$, it follows,

$$\Pr\{I_g = 1|K_i\} = \frac{1}{K_i - g} \cdot p_s(g)$$

where $1/(K_i - g)$ is the probability that the square (with side length $1/\sqrt{K_i - g}$) containing $s$ is selected in the $g$th iteration of Prim's algorithm, and $p_s$ is the probability that $s$ is selected in this square. Within this square which is further tessellated into cells with area $a(n)$, assume that $s$ is in the $p$th row and $q$th column, it follows,

$$p_s(g) = (p - 1)a^{\frac{3}{2}}(K_i - g)^{\frac{3}{2}} \cdot \left(\frac{1}{\sqrt{a(K_i - g)}} - p + 1\right)$$

$$+ (q - 1)a^{\frac{3}{2}}(K_i - g)^{\frac{3}{2}} \cdot \left(\frac{1}{\sqrt{a(K_i - g)}} - q + 1\right)$$

$$\leq 2\sqrt{a(n)(K_i - g)}$$

where the first (resp. second) term in (17) is the probability that $s$ lies in the same row (resp. column) as $u$ (resp. $v$). Therefore,

$$\Pr\{s \text{ selected by } U_i\} \leq \sum_{g=1}^{K_i-1} \Pr\{I_g = 1|K_i\}$$

$$\leq \sum_{g=1}^{K_i-1} \sum_{k=1}^{n+1} \frac{2\sqrt{a(n)(K_i - g)}}{k - g} \Pr\{K_i = k\}$$

$$\leq \sum_{k=1}^{n+1} 4\sqrt{2ka(n)} \Pr\{K_i = k\}$$

$$= \frac{4\sqrt{2a(n)}}{c(d)} |\text{EMST}(U)|$$

The lemma holds by setting $c_3 = 4\sqrt{2a(n)}/c(d)$. □

**Theorem 4.** Denote $N(s)$ as the number of multicast sessions that invoke $s$ for routing, then uniformly over all squarelets, it follows,

$$\lim_{n \to \infty} \Pr\left\{\cap_s \left\{N(s) \leq c_4 n |\text{EMST}(U)| \sqrt{a(n)}\right\}\right\} = 1$$

where $c_4$ is a positive constant.

**Proof.** Given an squarelet $s$, by definition:

$$N(s) = \sum_{i=1}^{n+1} 1\{s \text{ invoked by } U_i\}$$

where $1\{s \text{ invoked by } U_i\}$ are i.i.d. Bernoullian random variables with mean $p_1 \leq c_3 |\text{EMST}(U)| \sqrt{a(n)} = p_2$. Denote $N^*(s)$ as the corresponding sum of i.i.d. Bernoullian random variables with mean $p_2$, then clearly $N^*(s)$ is statistically larger than $N(s)$. By applying Chernoff bounds we get,

$$\Pr\{N(s) > 2\mathbb{E}[N^*(s)]\} < \Pr\{N^*(s) > 2\mathbb{E}[N^*(s)]\}$$

$$< (e/4)^{p_2} < e^{-np_2/8}$$

Since $|\text{EMST}(U)| \geq \Theta(1)$, $a(n) > \Theta(\log n/n)$,

$$\Pr\left\{\cap_s \left\{N(s) \leq 2c_4 n |\text{EMST}(U)| \sqrt{a(n)}\right\}\right\}$$

$$\geq 1 - \sum_s \Pr\{N(s) > 2\mathbb{E}[N^*(s)]\}$$

$$\geq 1 - ne^{-\sqrt{n}\log n/8} \to 1 \text{ as } n \to \infty$$

Setting $c_4 = 2c_3$ finishes the proof. □

Lastly, by employing a TDMA scheme as Section 3.3 such that every squarelet has a constant fraction of time to transmit, and further dividing a time slot into mini-slots such that a squarelet can deliver the traffic for every multicast session that invokes it, we have,

**Theorem 5.** The per-node throughput of the scheme for multicast is $\Omega\left(\frac{1}{n}|\text{EMST}(U)| \sqrt{a(n)}\right) = \cdots$
$$\Omega \left( \frac{1}{\sqrt{n \log n}} \right)$$, i.e.,

$$\begin{align*}
\Omega(1/\sqrt{n \log n}) & \quad \beta > 3/2 \\
\Omega(1/\log^{3/2} n) & \quad \beta = 3/2 \\
\Omega(n^{\beta - 2}/\sqrt{\log n}) & \quad 1 < \beta < 3/2 \\
\Omega(\log n/\sqrt{n}) & \quad \beta = 1 \\
\Omega(1/n) & \quad 0 \leq \beta < 1
\end{align*}$$

Corollary 2. The lower bound in Theorem 5 is tight up to a logarithmic factor.

Remark 3. Again, notice there is a small gap of $\sqrt{\log n}$ between the upper and lower bounds. This gap can be eliminated by modifying our communication scheme to base on percolation theory.

Remark 4. The results in Theorem 5 match well with Theorem 2 when $\alpha < 1$ and $\beta > 3/2$. For more sophisticated cases that $\alpha$ takes an arbitrary value, the multicast capacity is difficult to be obtained since calculating $\text{EMST}(U_i)$ appears intimidating. However, the nice consistency of the results in Theorem 5 with that in Theorem 2 helps us to conjecture a more general result for arbitrary $\alpha$.

Conjecture 1. If the number of destinations per multicast session follows power-law distribution with parameter $\beta$, and each destination is selected according to the rank-based model with parameter $\alpha$, then the multicast capacity is $\Theta(1/\sqrt{|D(X_i, Y_i)| \text{EMST}(U_i)\sqrt{n}})$.

6. CONCLUSION

This paper studies the throughput capacity of wireless networks with heterogeneous node distribution and relationship. We propose a simple model which captures the two key characteristics observed in real large scale networks, i.e., small-worldliness and power-law node degree distributions, and examine their impact on capacity. We show the fact that network heterogeneity leads to traffic locality and improves capacity in wireless networks. As well, in the common traffic pattern where information is desired to be disseminated from the source to all its contacts (e.g., friends, fans or followers) whose number follows power-law distribution, multicast may be employed to further enhance performance.

There are still many interesting directions for us to explore in the future. As is mentioned in previous sections, The network performance may be quite different from that obtained under static model. Furthermore, it is also interesting to take energy consumption and delay performance in mobile network into consideration. Because energy-efficiency and latency minimization are both hot topic in recent study of wireless networks.

7. REFERENCES


