Stochastic Differential Equation Theory Applied to Wireless Channels

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***Abstract*—Modeling wireless channels is essential to wireless communication systems. An autoregressive (AR) process of order one for wireless channel has long been assumed, but without a rigorous mathematical/physical basis. In this paper, we derive a first-order stochastic AR model for a flat stationary wireless channel, which comes from stochastic differential equation (SDE) theory concerning the nature of multipath fading channels. The resulting AR model describes more of the origin of multipath fading channels than previous AR models, and it can efficiently model and generate Rayleigh-distributed stationary fading channels. The Markovian property of the AR model is inherited through the SDE approach.**

***Index Terms*—Autoregressive (AR) process, multipath channels, Rayleigh, stochastic differential equation (SDE).**

I. Introduction

EXPERIMENTS with mobile communication at very high frequency (VHF) frequency began in the 1920s. The results of these experiments show that signal quality varies from "excellent" to "no signal." The signal incoming to the receiver contains a large number of reflected radio waves, which are characterized by "multipath reception." Thus, the wireless channel behaves in a random-like fashion.

The purpose of this paper is to develop a dynamical model of a stationary wireless channel in the case that the receiver is not moving, i.e., the zero Doppler shift. Jakes' description pertains to a wireless channel with Doppler shift and random component phases, but with deterministic temporal evolution. In contrast, we are concerned with the case that the component phases fluctuate in time, whereas in Jakes' model, they are assumed to be constant in time. For a stationary (i.e., zero Doppler) channel, Jakes' model results in a constant (one) as the autocorrelation that is inappropriate for long time periods because the stationary channel is still time-varying due to the relative phase fluctuations. This paper presents an SDE/AR-1 model to describe the dynamical behavior of such stationary channels. The SDE/AR-1 model is obtained through systematic mathematical reasoning starting from the (random walk) scattered electric-field model with multipath fading characterization. We are not concerned with nonstationary channels with varying Doppler shift.

The random behavior of a wireless channel can be viewed as a stochastic process. Stochastic differential equations (SDEs) are a powerful mathematical tool to analyze such processes. Traditional statistical analysis of probability distributed function (pdf), autocorrelation, etc., of stochastic processes cannot describe the random behavior of these in the time domain, in contrast to SDEs, which capture all the continuous-time statistical properties.

Several papers have been presented on the application of SDEs to the research of radar scattering and wireless communications. Field and Tough [4], [5] have successfully used SDEs to analyze K-distributed noise in electromagnetic scattering. Charalambous and Menemenlis [8] used SDEs to model multipath fading channels. In this paper, we will derive a simple SDE dynamical equation for the time variation of Rayleigh-distributed stationary wireless channels (cf. [4]). The underlying assumptions for our SDE model require Rayleigh-distributed stationary first-order Markovian multipath fading wireless channels. An extended dynamical description of electromagnetic propagation including the line of sight (LoS) reception, for which the received envelope consists of a superposition of specular and scattered components with resulting Ricean distribution, is provided *inter alia* in [15]. Rigorous mathematical analysis and computer simulation verify our SDE model. A first-order stochastic autoregressive (AR) model is derived directly by discretizing the SDE model in the time-variable.

There are two viewpoints that are traditionally taken in the literature. One is concerned with the mathematical tractability in models of wireless propagation. Another is concerned with the experimental accuracy and design aspects of the problem for which a higher order of the AR model may be more appropriate. In this paper, we adopt the former point of view, thus, preserving the Markov property and retaining mathematical simplicity and tractability. We derive a first-order AR process for a Rayleigh-distributed stationary wireless channel based on the SDE analysis of multipath Rayleigh fading. The SDE analysis and a resultant SDE model for multipath stationary Rayleigh fading channels are presented in Section II. A first-order AR model, as a discretization of the SDE, is discussed in Section III. The summary and conclusions are drawn in Section IV.

II. Wireless Channel SDE Model

Discussions on the modeling of multipath fading for stationary wireless channels to date tend to be limited in that no systematic analysis of a dynamical model is provided. Indeed, Jakes' model [see [10] and (A28)] does not describe the dynamics of a stationary channel, viewing its autocorrelation in the absence of Doppler as a constant, i.e., a time-invariant channel. In contrast, in what follows, we develop the time-dynamic model of a wireless channel for a stationary receiver, and obtain a first-order AR model representation based on systematic mathematical reasoning starting from the well-known scattered electric-field model with component phase fluctuations and multipath fading characterization as presented in (3).

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A. Modeling

The random walk model for the scattered electric field was developed in the 1980s (see [1]–[3]). These references give statistical description and correlation functions while no time dependent models were provided. This was extended in [4] and [5] to a continuous-time description in which the scattered electric field is modeled according to

$$\varepsilon_t = \sum \exp\left[i\varphi_t^{(k)}\right]. \tag{1}$$

The phase factors $\{\exp[i\varphi_t^{(k)}]\}$ are independent and uniformly distributed on the unit circle $\{|z| = 1, z \in \mathbb{C}\}$ ($\mathbb{C}$ is the set of all complex numbers), and $\varphi_t^{(k)}$ has uniform random initialization on the interval $[0, 2\pi)$ and satisfies

$$d\varphi_t^{(k)} = B^{1/2} dw_t^{(k)} \tag{2}$$

in which dw represents infinitesimal increments in a Wiener process w (see, e.g., [6]). Here, B is a constant with the dimension of frequency, which determines the correlation timescale for the component phase processes $\varphi_t^{(k)}$.

Considering a general wireless channel, the received signal on each path is random. Thus, we consider the following extended time description of the scattered electric field at the stationary receiver for a wireless channel with "multipath reception" characterization:

$$\varepsilon_t = \sum a_k \exp\left[i\varphi_t^{(k)}\right] \tag{3}$$

where the "form factor" a_k is the amplitude of the received signal on the kth path. The relative phase $\varphi_t^{(k)}$ satisfies the SDE (2). Comparing with Jakes' model (see [10] and also Appendix E), (3) introduces temporal relative-phase fluctuations without considering the Doppler shift in an explicit form. From (A4) (Ito's formula, see Appendix A), the Ito differential of (3) is (see Appendix B)

$$d\varepsilon_t^{(N)} = \sum_{k=1}^{N} a_k \left(i d\varphi_t^{(k)} - \frac{1}{2}\left(d\varphi_t^{(k)}\right)^2\right) \exp\left[i\varphi_t^{(k)}\right]. \tag{4}$$

When $N \to \infty$, if we introduce (2) into (4), after invoking the use of Ito's calculus [6], we arrive at (see Appendix C)

$$d\varepsilon_t^{(N)} = -\frac{1}{2} B \varepsilon_t^{(N)} dt + B^{\frac{1}{2}} \sigma d\xi_t \tag{5}$$

where $\sigma^2 = \sum_{k=1}^{N} a_k^2$, which is the variance of $\varepsilon_t^{(N)}$ (see Appendix D). An alternative derivation of the variance involves the computation of autocorrelation functions of in-phase and quadrature components that are computed at zero lag to give variance [14] but yield the same result. We will show that the variance of in-phase/quadrature components is $(1/2)\sigma^2$ in Section II-B [see (10)]. The quantity ξ_t is a complex Wiener process with properties (Appendix C)

$$|d\xi|^2 = dt \quad d\xi^2 = 0 \quad d\xi\, dt = 0. \tag{6}$$

For a real received signal $\varepsilon_t^{(N)}$, the variance of $\varepsilon_t^{(N)}$ is always finite for any N; hence, $\sigma < +\infty$ when $N \to \infty$. If we define $\Psi_t \triangleq \lim_{N\to\infty} \varepsilon_t^{(N)}$, then we can rewrite (5) as

$$d\Psi_t = -\frac{1}{2} B \Psi_t dt + B^{\frac{1}{2}} \sigma d\xi_t \tag{7}$$

where the random variable σ is the standard deviation, i.e., square root of the variance of Ψ_t. Thus, (7) is our SDE model for a wireless channel.

B. Solution of Wireless Channel SDE Model

We have presented a simple SDE model for a wireless channel as characterized by (3). In this section, we will give some mathematical analysis of the SDE model (7), and explore the relationship with the Rayleigh distribution.

Let the amplitude process Ψ_t of (7) be expressed in terms of its "in-phase" and "quadrature-phase" components I and Q

$$\Psi_t = I_t + iQ_t \tag{8}$$

where i is the square root of -1. Then I_t and Q_t can be described as two independent *Ornstein–Uhlenbeck* processes (e.g., [6]) satisfying the SDEs

$$\begin{aligned} dI_t &= -\frac{1}{2} B I_t\, dt + \frac{\sqrt{2}}{2} B^{1/2} \sigma\, dw_t^{(I)} \\ dQ_t &= -\frac{1}{2} B Q_t\, dt + \frac{\sqrt{2}}{2} B^{\frac{1}{2}} \sigma\, dw_t^{(Q)}. \end{aligned} \tag{9}$$

The pdf of I_t and Q_t have the same stationary forms

$$p(y) = \frac{1}{\sqrt{\pi}\sigma} \exp\left[-\frac{y^2}{\sigma^2}\right]. \tag{10}$$

Thus, I_t and Q_t are asymptotically Gaussian variables with mean zero and variance $(1/2)\sigma^2$. From (8) and (10), we find that the pdf of $|\Psi_t|$ ($\triangleq$ Z) will be a Rayleigh distribution, i.e.,

$$p(z) = \frac{2z}{\sigma^2} e^{-\frac{z^2}{\sigma^2}}. \tag{11}$$

Here, the basic fact that the square root of the sum of two squared Gaussian random variables is Rayleigh distributed has been used to obtain (11).

C. Model in Polar Form

In this section, we will explore the polar representation of the process Ψ_t for reasons that will become apparent in Section III. In this polar representation, the resultant phase fluctuations (16) provide a method to calculate the quantity B of (7) that is an essential ingredient in AR representation of the channel.

The complex amplitude process Ψ_t can alternatively be expressed in polar form

$$\Psi_t = R_t \exp(i\theta_t). \tag{12}$$

Thus, $i\theta_t = \log(\Psi_t/R_t)$. So, from Ito's formula (Appendix A) we have

$$i d\theta_t = \frac{d\Psi_t}{\Psi_t} - \frac{1}{2}\left(\frac{d\Psi_t}{\Psi_t}\right)^2 - \frac{dR_t}{R_t} + \frac{1}{2}\left(\frac{dR_t}{R_t}\right)^2. \tag{13}$$

Since the left-hand side is purely imaginary, we can express $d\theta_t$ in terms of Ψ_t alone [4] as

$$d\theta_t = \frac{1}{2i}\left[\left(\frac{d\Psi_t}{\Psi_t} - \frac{1}{2}\left(\frac{d\Psi_t}{\Psi_t}\right)^2\right) - \left(\frac{d\Psi_t^*}{\Psi_t^*} - \frac{1}{2}\left(\frac{d\Psi_t^*}{\Psi_t^*}\right)^2\right)\right]. \tag{14}$$

From (6) and (7), we have

$$\frac{d\Psi_t}{\Psi_t} - \frac{1}{2}\left(\frac{d\Psi_t}{\Psi_t}\right)^2 = -\frac{1}{2}B\,dt + \frac{B^{1/2}\sigma}{\Psi_t}d\xi_t. \tag{15}$$

It is to be noted that $(\frac{d\Psi_t}{\Psi_t})^2 = 0$ because of (6) and (7).

Thus, substituting (15) into (14), we get

$$d\theta_t = \left(\frac{B\sigma^2}{2z_t}\right)^{1/2} dw_t^{(\theta)} \tag{16}$$

for the resultant phase fluctuations where $z_t = |\Psi_t|^2$, which is the intensity process introduced ealier, and $dw_t^{(\theta)}$ is a new Wiener process defined by

$$dw_t^{(\theta)} = \frac{1}{i\sqrt{2}\,|\Psi_t|}\left(\Psi_t^* d\xi_t - \Psi_t\, d\xi_t^*\right). \tag{17}$$

Thus, the angular process θ_t has a zero-drift component for a stationary Rayleigh-distributed wireless channel.

III. AR Model for Wireless Channel

In this section, we will explore the close relationship between the SDE and AR channel representations.

A. Derivation of AR Model

Let us consider discrete-time samplings for the SDE process in (7) with equally spaced sample times $t_0, t_1, \ldots, t_k, \ldots$ where $t_{k+1} - t_k = \delta t$. After replacing the continuous-time differential $d\Psi_t$ in (7) with the discrete-time difference $(\Psi_{t_{k+1}} - \Psi_{t_k})$, replacing Ψ_t with the arithmetic average $(\Psi_{t_{k+1}} + \Psi_{t_k})/2$, and replacing the Wiener process $d\xi_t$ with a complex (indicated by tilde) discrete Gaussian process $\delta t^{1/2}\tilde{n}(k)$, we obtain

$$\Psi_{t_{k+1}} = \frac{1 - \frac{1}{4}B\,\delta t}{1 + \frac{1}{4}B\,\delta t}\Psi_{t_k} + \frac{B^{\frac{1}{2}}\sigma\,\delta t^{1/2}}{1 + \frac{1}{4}B\,\delta t}\tilde{n}(k) \tag{18}$$

where $\tilde{n}(k)$ is a standard complex Gaussian process with zero mean and unit variance. Equation (18) is a first-order AR process for the wireless channel. The AR coefficients are functions of the constant B, the sampling time interval δt, and the square root of variance of channel σ.

The multipath Rayleigh scattering model (3) has been used for stationary multipath fading wireless channels for a long time. Traditionally, an AR process of order one has been assumed in the Rayleigh scattering model. However, in the literature to date, there has not been a rigorous systematic derivation of the dynamics, which is where the proposed SDE theory becomes an essential extra ingredient. For the traditional AR model, there is no account given of the direct relationship between the AR coefficients and the underlying physical characteristics of multipath wireless channels, i.e., a resultant amplitude arising as a superposition of a number of random phasors, each evolving on a suitable time scale. The AR model developed in this paper is derived from these basic physical principles using the techniques from SDE theory to describe the continuous time evolution, and then, passing to discrete time. Our development makes transparent the relations between the AR coefficients and the physical [correlation timescale] parameter B, the variance of the channel, and the discrete sampling interval, as demonstrated in (18). The physical origins of the AR coefficients are, thus, determined. By comparison, the AR coefficients can also be obtained [12] statistically using Yule–Walker equations and autocorrelation of the process.

To simulate a Rayleigh-distributed channel, (3) requires large N (thousands or more) paths, which results in a lot of computation. Instead of simulating these large N paths, we model the total behavior of all these paths by one stochastic process Ψ_t. To generate a single value of channel data at instant time t, we need to produce N random number $\varphi_t^{(j)}$ and a_j from (3), while only one complex random number $\delta\xi_t$ (two real random numbers) from (18). Thus, the computational complexity of the SDE model (7) is around $1/N$ of (3). Considering the large N, the SDE model provides a very simple way to simulate Rayleigh-distributed wireless channels more efficiently.

Wong and Chang verified experimentally the first-order Markovian model for a Rayleigh fading channel in [7]. They concluded that the first-order Markovian chain is sufficient to model a Rayleigh fading channel for any application. The first-order Markovian assumption implies that, given the information of current state, any future state should be independent of the previous state. Our first-order AR model (discrete-time SDE) (18) derived from the essential scattered electrical field equation (multipath) (3) complies with the first-order Markovian assumption. There is evidence for improving the model accuracy by increasing the model order [11], [13]. Here, we restrict ourselves to an AR model of order one. The rationale behind this is as follows. The SDE theory favors the dynamical description of a wireless channel for a stationary receiver, as developed in Section II. Moreover, its discretization is automatically an AR model of order one, and it preserves the Markovian property since the underlying SDE is Markov in this case. Jakes' model overlooks the statistical analysis of stationary channels (i.e., channels without Doppler shift). The familiar Bessel function autocorrelation $J_0(2\pi f_D \tau)$ obtained from Jakes' model results in a constant (one) for a stationary channel; thus, Jakes' model is inappropriate to describe a stochastic process that we consider here in which the temporal channel behavior is due to relative phase fluctuations. Although second- or higher order AR models may be appropriate to obtain closer approximation to the autocorrelation of a wireless channel with a particular Doppler frequency, we are not concerned with such nonstationary wireless channels here. We present a novel approach to obtaining a first-order AR model for a stationary wireless channel through systematic mathematical reasoning from first principles, beginning with the well-accepted multipath scattered electric-field

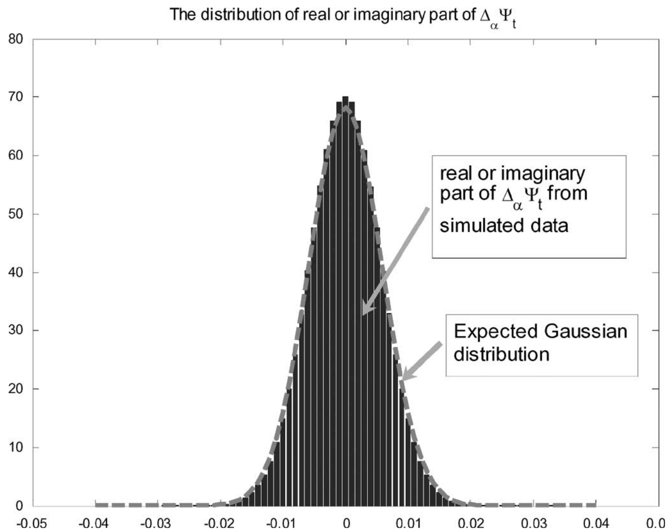

Fig. 1. Comparison of the difference in process from simulated channel data with the expected theoretical Gaussian distribution. Simulated channel data is generated with $N = 50$, $B = 202$ s^{-1}, sampling time $\delta t = 10$E-6 s, and $\sigma^2 = 0.32$.

model, i.e., (3). Simulated data will be shown to verify our AR model in Section III-B.

B. Verifying the AR Model Using Simulated Data

With computer-generated channel data Ψ_{t_k} from (3) and (2), we define a difference process as

$$\Delta_\alpha \Psi_{t_k} = \Psi_{t_{k+1}} - \alpha \Psi_{t_k} \tag{19}$$

where

$$\alpha = \frac{1 - \frac{1}{4} B\,\delta t}{1 + \frac{1}{4} B\,\delta t}. \tag{20}$$

If we can verify that the difference process $\Delta_\alpha \Psi_{t_k}$ is a complex Gaussian process with zero mean and variance β^2, where

$$\beta = \frac{B^{\frac{1}{2}} \sigma\, \delta t^{1/2}}{1 + \frac{1}{4} B\,\delta t} \tag{21}$$

then, the AR process for a wireless channel (18) will be verified.

Fig. 1 verifies that the real and imaginary parts of the difference process $\Delta_\alpha \Psi_{t_k}$ are the Gaussian processes that we expected. Thus, our derived AR process described in (18) is well verified.

IV. Conclusion

We have presented a first-order stochastic AR model for a wireless channel, which is based on SDE modeling of stationary Rayleigh fading wireless channels with "multipath reception" characterization. A rigorous and principled mathematical derivation of the origin of our stochastic AR model has been provided in detail. The AR model provides more channel information than previous AR models for stationary wireless channels. Moreover, simulated wireless channel data lend strong support to our first-order stochastic AR model.

The essential features of the proposed stochastic AR model can be summarized as follows.

1) It can model stationary Rayleigh-distributed fading channels effectively.
2) It can efficiently generate synthetic Rayleigh-distributed channel data.
3) It is an instance of a first-order Markov chain.
4) The model developed follows from the SDE theory under assumptions concerning the nature of multipath fading channels.
5) The AR parameters express physical meanings.

Future work will address the dynamics of the extended random walk model as in (A28) from an SDE point of view, including the effects of Doppler and allowing the relative phases $\varphi_t^{(k)}$ to *fluctuate* in time. In this respect, the dynamical description and corresponding autocorrelation/spectral analysis for fixed Doppler shift independent of the path has been obtained previously by Field and Tough in [4], and will motivate this subsequent line of development.

Appendix A
Ito's Formula

We provide an informal sketch of the proof of Ito's formula here. Suppose, x_t is an Ito process that obeys a stochastic differential equation

$$dx_t = b\,dt + c\,dw_t. \tag{A1}$$

Let $f(t, x_t)$ be a function of t and x_t, and consider the differential of $F_t = f(t, x_t)$. Observe that (cf., Taylor's theorem)

$$dF_t = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x_t} dx_t + \frac{1}{2}\frac{\partial^2 f}{\partial x_t^2} dx_t^2 + \frac{1}{2}\frac{\partial^2 f}{\partial t^2} dt^2 + \frac{\partial^2 f}{\partial x_t \partial t} dx_t dt + \dots \tag{A2}$$

We also know [6]

$$\begin{aligned} dw_t^2 &= dt \\ dw_t\,dt &= 0 \\ dt^2 &= 0. \end{aligned} \tag{A3}$$

Substituting (A1) into (A2) with (A3) yields only the first three nonzero terms of (A2) as

$$dF_t = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x_t} dx_t + \frac{1}{2}\frac{\partial^2 f}{\partial x_t^2} dx_t^2. \tag{A4}$$

In essence, we only take the first three terms on the right-hand side of (A2) to obtain this result, since $dw_t^2 = dt$, and terms of $O(dt^\alpha), \alpha > 1$ contribute zero to stochastic integrals [6]. Substituting (A1) into (A4) with (A3) yields Ito's formula

$$dF_t = \left(\frac{\partial f}{\partial t} + b\frac{\partial f}{\partial x_t} + \frac{c^2}{2}\frac{\partial^2 f}{\partial x_t^2} \right) dt + c\frac{\partial f}{\partial x_t} dw_t. \tag{A5}$$

Appendix B
Proof of the SDE (4)

We provide a proof of the SDE (4). Corresponding to (A1) and Ito's formula (A5), the notations used in the paper are as follows. The phase $\varphi_t^{(k)}$ in (2) corresponds to x_t in (A1). Thus, comparing (2) and (A1), we have $b = 0$ and $c = B^{1/2}$. The ε_t in (3) corresponds to F_t in (A4), and identifying ε_t with $F_t = f(t, x_t)$ we obtain

$$\begin{aligned} dx_t &= d\varphi_t^{(k)} \\ \frac{\partial f}{\partial t} &= 0 \\ \frac{\partial f}{\partial x_t} &= \frac{\partial\left(\sum a_k \exp\left[i\varphi_t^{(k)}\right]\right)}{\partial\varphi_t^{(k)}} = ia_k \exp\left[i\varphi_t^{(k)}\right] \\ \frac{\partial^2 f}{\partial x_t^2} &= \frac{\partial\left(ia_k \exp\left[i\varphi_t^{(k)}\right]\right)}{\partial\varphi_t^{(k)}} = -a_k \exp\left[i\varphi_t^{(k)}\right]. \end{aligned} \quad \text{(A6)}$$

Notice that the index k contributes to the summation in (3); thus, (A4) is extended to the multivariate formula for the collection of component phases $\varphi_t^{(k)}, k = 1, 2, \ldots$ as

$$dF_t = \frac{\partial f}{\partial t}dt + \sum \frac{\partial f}{\partial\varphi_t^{(k)}} d\varphi_t^{(k)} + \frac{1}{2}\sum \frac{\partial^2 f}{\partial\left(\varphi_t^{(k)}\right)^2}\left(d\varphi_t^{(k)}\right)^2. \quad \text{(A7)}$$

Now, substituting all the relations in (A6) into (A7) yields

$$\begin{aligned} d\varepsilon_t &= \sum ia_k \exp\left[i\varphi_t^{(k)}\right] d\varphi_t^{(k)} - \frac{1}{2}\sum a_k \exp\left[i\varphi_t^{(k)}\right]\left(d\varphi_t^{(k)}\right)^2 \\ &= \sum a_k \left(id\varphi_t^{(k)} - \frac{1}{2}\left(d\varphi_t^{(k)}\right)^2\right) \exp\left[i\varphi_t^{(k)}\right] \end{aligned}$$

and thus, (4) is obtained.

Appendix C

Proof of the SDE (5)

Assuming $dw_t^{(j)}$ and $dw_t^{(k)}$ for any j, k $(j \neq k)$ are independent. Introducing (2) into (4) and applying $(dw_t^{(j)})^2 = dt$, we have

$$\begin{aligned} d\varepsilon_t^{(N)} &= \sum_{j=1}^N a_j \left(id\varphi_t^{(j)} - \frac{1}{2}\left(d\varphi_t^{(j)}\right)^2\right) \exp\left[i\varphi_t^{(j)}\right] \\ &= \sum_{j=1}^N a_j \left(iB^{1/2} dw_t^{(j)} - \frac{1}{2}Bdt\right) \exp\left[i\varphi_t^{(j)}\right] \\ &= \sum_{j=1}^N a_j iB^{1/2} dw_t^{(j)} \exp\left[i\varphi_t^{(j)}\right] \\ &\quad - \frac{1}{2}Bdt\sum_{j=1}^N a_j \exp\left[i\varphi_t^{(j)}\right] \\ &= V - \frac{1}{2}B\varepsilon_t^{(N)}dt \end{aligned} \quad \text{(A8)}$$

where

$$V = \sum_{j=1}^N a_j iB^{1/2} dw_t^{(j)} \exp\left[i\varphi_t^{(j)}\right]. \quad \text{(A9)}$$

Furthermore, we decompose V into real and imaginary parts as

$$\begin{aligned} V &= B^{1/2}\left[\sum_{j=1}^N a_j i\cos\left(\varphi_t^{(j)}\right) dw_t^{(j)} - \sum_{j=1}^N a_j \sin\left(\varphi_t^{(j)}\right) dw_t^{(j)}\right] \\ &= B^{1/2}\left[i\sum_{j=1}^N a_j \cos\left(\varphi_t^{(j)}\right) dw_t^{(j)} - \sum_{j=1}^N a_j \sin\left(\varphi_t^{(j)}\right) dw_t^{(j)}\right]. \end{aligned} \quad \text{(A10)}$$

The quantities $\sum_{j=1}^N a_j \cos(\varphi_t^{(j)}) dw_t^{(j)}$ and $\sum_{j=1}^N a_j \sin(\varphi_t^{(j)}) dw_t^{(j)}$ appearing in (A10) are scaled Wiener processes. We have

$$\sum_{j=1}^N a_j \cos\left(\varphi_t^{(j)}\right) dw_t^{(j)} = \sigma_c dw_t^{(c)} \quad \text{(A11)}$$

where $\sigma_c^2 = \sum_{j=1}^N a_j^2 \cos^2(\varphi_t^{(j)})$, and

$$\sum_{j=1}^N a_j \sin\left(\varphi_t^{(j)}\right) dw_t^{(j)} = \sigma_s dw_t^{(s)} \quad \text{(A12)}$$

where $\sigma_s^2 = \sum_{j=1}^N a_j^2 \sin^2(\varphi_t^{(j)})$.

In (A11) and (A12), we claim that $dw_t^{(c)}$ and $dw_t^{(s)}$ are independent Wiener processes. Proof of independence follows.

Proof: From (A10)–(A12), we have

$$\begin{aligned} V &= B^{1/2}\left[i\sigma_c\, dw_t^{(c)} - \sigma_s\, dw_t^{(s)}\right] \\ &= B^{1/2}\sigma\left[i\frac{\sigma_c}{\sigma} dw_t^{(c)} - \frac{\sigma_s}{\sigma} dw_t^{(s)}\right] \end{aligned} \quad \text{(A13)}$$

where

$$\sigma^2 = \sigma_c^2 + \sigma_s^2 = \sum_{j=1}^N a_j^2. \quad \text{(A14)}$$

We have

$$\begin{aligned} &\left(\frac{\sigma_c}{\sigma} dw_t^{(c)}\right)\left(\frac{\sigma_s}{\sigma} dw_t^{(s)}\right) \\ &= \frac{1}{2\sigma^2}\sum_{j=1}^N a_j^2 \sin\left(2\varphi_t^{(j)}\right) dt \\ &= \frac{1}{2}dt\sum_{j=1}^N \alpha_j \sin\left(2\varphi_t^{(j)}\right) \end{aligned} \quad \text{(A15)}$$

where $\alpha_j = a_j^2/\sigma^2$ and $\sum_{j=1}^N \alpha_j = 1$.

For $N \to \infty, \alpha_j \to 0+$, let $\{\varphi_t^{(\rho(j))}\}$ be a permutation of $\{\varphi_t^{(j)}\}$ so that $0 \leq \varphi_t^{(\rho(1))} < \varphi_t^{(\rho(2))} < \cdots < \varphi_t^{(\rho(N))} \leq 2\pi$. Then, in (A15) we have

$$\begin{aligned} \sum_{j=1}^N \alpha_j \sin\left(2\varphi_t^{(j)}\right) &= \sum_{j=1}^N \alpha_j \sin\left(2\varphi_t^{(\rho(j))}\right) \\ &\simeq \frac{1}{2\pi}\int_0^{2\pi} \sin(2x)\, dx \\ &= 0. \end{aligned} \quad \text{(A16)}$$

From (A15) and (A16), we have

$$\left(\frac{\sigma_c}{\sigma} dw_t^{(c)} z\right)\left(\frac{\sigma_s}{\sigma} dw_t^{(s)}\right) = 0, \qquad \text{for } N \to \infty. \quad \text{(A17)}$$

Thus, $dw_t^{(c)}$ and $dw_t^{(s)}$ are independent. Here, we use the fact that if $dw_t^{(c)}\, dw_t^{(s)} = 0, dw_t^{(c)}$ and $dw_t^{(s)}$ are independent [9].

Furthermore, we have

$$\begin{aligned}\left(\frac{\sigma_c}{\sigma}\right)^2 &= \frac{1}{\sigma^2}\sum_{j=1}^{N} a_j^2 \cos^2\left(\varphi_t^{(j)}\right)\\ &= \sum_{j=1}^{N} \alpha_j \cos^2\left(\varphi_t^{(j)}\right)\\ &= \frac{1}{2} + \frac{1}{2}\sum_{j=1}^{N}\alpha_j \cos\left(2\varphi_t^{(j)}\right) \qquad \text{(A18)}\\ &= \frac{1}{2}\,(N \to \infty).\end{aligned}$$

In (A18), we use the fact that

$$\sum_{j=1}^{N}\alpha_j \cos\left(2\varphi_t^{(j)}\right) \overset{N\to\infty}{=} \frac{1}{2\pi}\int_0^{2\pi}\cos(2x)dx = 0 \quad \text{(A19)}$$

Similarly, we have

$$\left(\frac{\sigma_s}{\sigma}\right)^2 \overset{N\to\infty}{=} \frac{1}{2}. \quad \text{(A20)}$$

Introducing (A18) and (A20) into (A13), we have

$$V = \frac{B^{1/2}\sigma}{\sqrt{2}}\left[i dw_t^{(c)} - dw_t^{(s)}\right] = B^{1/2}\sigma\, d\xi_t \quad \text{(A21)}$$

where

$$d\xi_t = \frac{1}{\sqrt{2}}\left(i dw_t^{(c)} - dw_t^{(s)}\right). \quad \text{(A22)}$$

Since $dw_t^{(c)}$ and $dw_t^{(s)}$ are independent Wiener processes, $d\xi_t$ is a complex Wiener process. From (A22), we deduce the following properties of $d\xi_t$:

$$|d\xi|^2 = dt \qquad d\xi^2 = 0. \quad \text{(A23)}$$

Now, from (A8) and (A21), and for ξ_t in (A23), we have shown

$$d\varepsilon_t^{(N)} = -\frac{1}{2}B\varepsilon_t^{(N)}dt + B^{1/2}\sigma\, d\xi_t \quad \text{(A24)}$$

that is, (5).

Appendix D

From the definition of $\varepsilon_t^{(N)}$ in (3), we have

$$\begin{aligned} var\left(\varepsilon_t^{(N)}\right) &= E\left(\left|\varepsilon_t^{(N)} - E\left(\varepsilon_t^{(N)}\right)\right|^2\right) = E\left(\varepsilon_t^{(N)}\left(\varepsilon_t^{(N)}\right)^*\right)\\ &= \sum_{k=1}^{N}\sum_{j=1}^{N} a_j a_k E\left(\exp\left[i\varphi_t^{(j)} - i\varphi_t^{(k)}\right]\right). \quad \text{(A25)}\end{aligned}$$

For $j \neq k, E(\exp[i\varphi_t^{(j)} - i\varphi_t^{(k)}]) = 0$. Equation (A25) is, thus, simplified as

$$\text{var}\left(\varepsilon_t^{(N)}\right) = \sum_{j=1}^{N} a_j^2. \quad \text{(A26)}$$

From (A14) and (A26), we have

$$\sigma^2 = \text{var}\big(\varepsilon_t^{(N)}\big). \quad \text{(A27)}$$

See main text under (5).

Appendix E
Jakes' Model

Consider a moving receiver. The mathematical form of Jakes' channel model, which includes the effects of Doppler shift is an extension of the random walk model as (3)

$$\varepsilon_t = \sum_{k=1}^{N} a_k \exp\left[i\big(2\pi f_k t + \varphi^{(k)}\big)\right] \quad \text{(A28)}$$

where a_k is the electric-field strength of the kth path, the $\varphi^{(k)}$ is its relative phase, and f_k is its Doppler shift. Notice that in Jakes' model, the relative phases $\varphi^{(k)}$ are assumed *constant* in time. The quantity ε_t represents the complex envelope of N signal rays assuming that all rays are arriving from a horizontal direction and come from arbitrary angles surrounding the receiver.

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