Finite-SNR Diversity-Multiplexing Tradeoff for Two-Way Relay Fading Channel

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Abstract—This paper studies the performance limits of two-way relay channel (TWRC) at finite signal-to-noise ratio (SNR) in Rayleigh fading environment. A two-phase decode-and-forward (DF) protocol is considered. We first derive closed-form expressions for both outage probability and diversity-multiplexing tradeoff (DMT). Our results are general and suitable for any time sharing and any rate allocation in the two-way relay protocol. It is found that DF outperforms amplify-and-forward (AF) when either multiplexing gain or SNR is small enough, otherwise, DF is inferior to AF in term of outage probability. Meanwhile, finite-SNR DMT of DF is always lower than that of AF regardless of SNR due to the additional sum-rate constraint imposed on the relay node for full decoding. Furthermore, the optimum relay location for any given combination of time sharing and rate allocation is presented.

I. INTRODUCTION

Thanks to its great potential in enhancing spectral efficiency, two-way relay channel (TWRC), where two source nodes exchange information with the help of a relay node, has attracted a lot of interests [1]–[7]. Several relay protocols have been proposed, including two-phase amplify-and-forward (AF) [4], two-phase and three-phase decode-and-forward (DF) [5] [6] [7]. A common performance metrics applied in these works to evaluate these protocols is achievable rate region, which however is for deterministic channels. For wireless fading channels, new performance metrics are desired for further understanding of the performance limits of two-way relaying.

Diversity-multiplexing tradeoff (DMT), firstly proposed by Zheng and Tse [8] has become a fundamental and essential tool to analyze the performance of not only multiple-input multiple-output (MIMO) fading channels but also relay fading channels. Recently, the authors in [9] and [10] present the DMT analysis for both half-duplex and full-duplex TWRC with DF and compress-and-forward protocols. It is noted that these studies are for asymptotically high SNR region. However, practical communication systems, such as wireless local area network and cellular networks usually operate in the low and medium SNR range. More importantly, it is only in the low and medium SNR region that relay-assisted transmission will play an important role (if SNR is high enough, direct transmission is preferred). Hence, it is more desirable to

Fig. 1. System model of TWRC

analyze DMT at finite SNR range as presented in [11]. Given that, for quasi-static fading channels, the packet error rate is equal to the outage probability with the capacity-achieving codes applied, the authors in [11] present the new definitions of diversity gain and multiplexing gain at finite SNR. The multiplexing gain determines how the target data rate scales with a finite SNR, while diversity gain at a particular SNR is used to predict the additional SNR required to decrease a specified amount of outage probability for a given multiplexing gain. Recently, authors in [12] extend the finite-SNR DMT analysis for TWRC. But it is for AF protocol only.

In this paper, we complement the literature of DMT analysis for TWRC at finite SNR by considering two-phase DF protocol. Closed-form expressions for both outage probability and finite-SNR DMT in Rayleigh fading channels are derived. Note that the priori work in [12] only considers equal time sharing and fair rate allocation. Our results are general and suitable for any time sharing and rate allocation in the two-way relay protocol. Through numerical study, the optimum relay location under certain conditions on time sharing and rate allocation is discussed as well.

II. SYSTEM MODEL

Consider a TWRC shown in Fig. 1, where two sources S_1 and S_2 communicate with each other aided by the relay R . It is supposed that every terminal is equipped with single antenna and has the same transmission power E (the extension to different power is straightforward). Each link is assumed to be reciprocal, i.e., the channel coefficients between S_1 and R , and between S_2 and R satisfy $h_{1R} = h_{R1} = h$, $f_{2R} = f_{R2} = f$. Here both h and f are zero-mean complex Gaussian random variables with variances Ω_h and Ω_f , respectively.

The bidirectional transmission takes place in two phases using DF relay protocol. In the first phase, also called multiple-access (MA) phase, S_1 and S_2 transmit simultaneously towards R for a time fraction of t and the relay attempts to jointly

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decode the two signals. Assume the noise in all links is additive Gaussian with zero mean and unit variance. Then, the instantaneous SNRs of the links between the two source nodes and the relay node can be denoted as $\gamma_{1R} = E|h|^2$ and $\gamma_{2R} = E|f|^2$, respectively. The instantaneous total received SNR at R is $\gamma_{\text{mac}} = E|h|^2 + E|f|^2$. To successfully decode both signals from two sources at the relay, the achievable rate pair (R_1, R_2) of the two sources needs to satisfy the following MA channel constraints:

$$R_1 \leq I_{1R} \triangleq t \log_2(1 + \gamma_{1R}), \quad (1)$$

$$R_2 \leq I_{2R} \triangleq t \log_2(1 + \gamma_{2R}), \quad (2)$$

$$R_1 + R_2 \leq I_{\text{mac}} \triangleq t \log_2(1 + \gamma_{\text{mac}}). \quad (3)$$

In the second phase, named as broadcast (BC) phase, the relay R, after applying ideal random binning, transmits the combined signals towards the two desired destinations for a time fraction of $1 - t$. Note that the total transmission time of the considered two-phase protocol is normalized to one and we refer to t as the time-sharing parameter. In the BC phase, to ensure successful decoding at both terminals, the achievable rate pair (R_1, R_2) needs to satisfy [6]:

$$R_1 \leq I_{R2} \triangleq (1 - t) \log_2(1 + \gamma_{2R}), \quad (4)$$

$$R_2 \leq I_{R1} \triangleq (1 - t) \log_2(1 + \gamma_{1R}). \quad (5)$$

Applying the cut-set bound theory, the achievable rate region of TWRC is then given by all the possible rate pairs (R_1, R_2) satisfying:

$$R_1 \leq \min(I_{1R}, I_{R2}), \quad (6)$$

$$R_2 \leq \min(I_{2R}, I_{R1}), \quad (7)$$

$$R_1 + R_2 \leq I_{\text{mac}}. \quad (8)$$

III. OUTAGE PROBABILITY

To obtain finite-SNR DMT, we derive the outage probability first in this section. Let R denote the target sum-rate of TWRC. The target rates of source S_1 and S_2 are αR and $(1 - \alpha)R$, respectively, where α is called the rate allocation parameter. The TWRC system is said to be in outage when any source transmission is in outage. In response to the rate constraints (6)-(8), we define three events:

$$\begin{cases} A_1 : \alpha R \leq \min(I_{1R}, I_{R2}) \\ A_2 : (1 - \alpha)R \leq \min(I_{2R}, I_{R1}) \\ A_3 : R \leq I_{\text{mac}} \end{cases} \quad (9)$$

Then, the outage probability of the TWRC system can be expressed as:

$$P_{\text{out}} = 1 - P(A_1 \cap A_2 \cap A_3). \quad (10)$$

Substituting the definitions of I_{iR} , I_{Ri} and I_{mac} , for $i = 1, 2$ in (1)-(5) into the events (9) and after simple manipulation, we obtain the equivalent conditions on the channel gain pair

Fig. 2. Channel gain pair (X, Y)

$(X \triangleq |h|^2, Y \triangleq |f|^2)$ for reliable communications as

$$\begin{cases} X \geq \max\left(\frac{2^{\frac{\alpha R}{t}} - 1}{E} \triangleq C_{X1}, \frac{2^{\frac{(1-\alpha)R}{1-t}} - 1}{E} \triangleq C_{X2}\right) \\ Y \geq \max\left(\frac{2^{\frac{\alpha R}{1-t}} - 1}{E} \triangleq C_{Y1}, \frac{2^{\frac{(1-\alpha)R}{t}} - 1}{E} \triangleq C_{Y2}\right) \\ X + Y \geq \frac{2^{\frac{R}{t}} - 1}{E} \triangleq C_0 \end{cases} \quad (11)$$

Here, both channel gains X and Y follow exponential distribution with mean Ω_h and Ω_f , respectively. In (11), both time-sharing parameter t and rate allocation parameter α are considered. In the following three subsections, we derive the probability of the joint event (11) and obtain the outage probability when both parameters take different values.

A. Equal time sharing and arbitrary rate allocation

For simplicity, we first assume equal time sharing, i.e., $t = 1/2$. Thus, (11) becomes $\{(X, Y) \mid X \geq \max(\frac{2^{2\alpha R} - 1}{E} \triangleq C_1, \frac{2^{2(1-\alpha)R} - 1}{E} \triangleq C_2), Y \geq \max(C_1, C_2), X + Y \geq \frac{2^{2R} - 1}{E} \triangleq C_0^*\}$. Without loss of generality, we let the target rate of S_1 is greater than or equal to the target rate of S_2 , i.e., $1/2 \leq \alpha \leq 1$. Thus $C_1 \geq C_2$. Hence, after further simplification, (X, Y) satisfies $\{(X, Y) \mid X \geq C_1, Y \geq C_1, X + Y \geq C_0^*\}$.

Depending on the value of C_0^* in comparison with C_1 , the region of the channel gain pair (X, Y) that satisfies the above conditions can be plotted in Fig. 2, as shown by shadowed area. Thus, using the probability distribution function of exponential variables and considering that X and Y are independent, we can obtain the outage probability of the TWRC system as:

$$P_{\text{out}}(\alpha, t=1/2) = \begin{cases} 1 - e^{-\left(\frac{1}{\Omega_h} + \frac{1}{\Omega_f}\right)C_1}, & \alpha_0 \leq \alpha \leq 1 \\ 1 - \frac{\Omega_h}{\Omega_h - \Omega_f} e^{-\frac{1}{\Omega_h}C_0^* + \left(-\frac{1}{\Omega_f} + \frac{1}{\Omega_h}\right)C_1} \\ \quad + \frac{\Omega_f}{\Omega_h - \Omega_f} e^{-\frac{1}{\Omega_f}C_0^* + \left(-\frac{1}{\Omega_h} + \frac{1}{\Omega_f}\right)C_1}, & \frac{1}{2} \leq \alpha \leq \alpha_0 \end{cases} \quad (12)$$

where $\alpha_0 \triangleq (\log_2(2^{2R} + 1) - 1)/2R$ is the root of $2C_1 = C_0^*$, i.e., $2 \cdot (2^{2\alpha R} - 1) = 2^{2R} - 1$, and is shown to be greater than or equal to $1/2$. In the special case where the two channel statistics are the same, i.e., $\Omega_h = \Omega_f$, the outage probability

is:

$$P_{\text{out}(\alpha, t=1/2)} = \begin{cases} 1 - e^{-\frac{2}{\Omega_h} C_1}, & \alpha_0 \leq \alpha \leq 1 \\ 1 - e^{-\frac{1}{\Omega_h} C_0^*} \\ -\frac{1}{\Omega_h} (C_0^* - 2C_1) e^{-\frac{1}{\Omega_h} C_0^*}, & \frac{1}{2} \leq \alpha \leq \alpha_0 \end{cases} \quad (13)$$

B. Arbitrary time sharing and equal rate allocation

Next, let the rate allocation be fair, i.e., the target rate of each source is $R/2$. Then, (11) is simplified as $\{(X, Y) \mid X \geq \max(\frac{2^{\frac{R}{2t}} - 1}{E} \triangleq C_3, \frac{2^{\frac{R}{2(1-t)}} - 1}{E} \triangleq C_4), Y \geq \max(C_3, C_4), X + Y \geq C_0\}$.

1) For $t \leq \frac{1}{2}$: we have $C_3 \geq C_4$, and channel gain pair (X, Y) is in the set of $\{(X, Y) \mid X \geq C_3, Y \geq C_3, X + Y \geq C_0\}$. This is similar to Fig. 2, except that C_1 and C_0^* are replaced by C_3 and C_0 , respectively. Since $2C_3 \leq C_0$, i.e., $2 \cdot (2^{\frac{R}{2t}} - 1) \leq 2^{\frac{R}{t}} - 1$ always holds if $0 \leq t \leq \frac{1}{2}$, only the event shown in Fig. 2(b) will occur. Thus, the outage probability of the system is given by:

$$P_{\text{out}(\alpha=1/2, t)} = 1 - \frac{\Omega_h}{\Omega_h - \Omega_f} e^{-\frac{1}{\Omega_h} C_0 + (-\frac{1}{\Omega_f} + \frac{1}{\Omega_h}) C_3} + \frac{\Omega_f}{\Omega_h - \Omega_f} e^{-\frac{1}{\Omega_f} C_0 + (-\frac{1}{\Omega_h} + \frac{1}{\Omega_f}) C_3}. \quad (14)$$

If $\Omega_h = \Omega_f$, the outage probability of the system is:

$$P_{\text{out}(\alpha=1/2, t)} = 1 - e^{-\frac{1}{\Omega_h} C_0} - \frac{1}{\Omega_h} (C_0 - 2C_3) e^{-\frac{1}{\Omega_h} C_0}. \quad (15)$$

2) For $t > \frac{1}{2}$: we have $C_3 < C_4$. Hence, channel gain pair (X, Y) satisfies $\{(X, Y) \mid X \geq C_4, Y \geq C_4, X + Y \geq C_0\}$, as in Fig. 2, with C_1 and C_0^* replaced by C_4 and C_0 , respectively. In the same way, channel gain pair (X, Y) is in Fig. 2(a) when $2C_4 > C_0$, i.e., $2 \cdot (2^{\frac{R}{2(1-t)}} - 1) > 2^{\frac{R}{t}} - 1$, otherwise, in Fig. 2(b). This inequality is transcendental which does not have exact analytical solution. However, note that the function $2^{\frac{R}{t}} - 2 \cdot 2^{\frac{R}{2(1-t)}} + 1$ at a given R is monotonically decreasing with respect to t . So there exists a unique zero and we denote it as t_0 . Hence, the outage probability of the system is:

$$P_{\text{out}(\alpha=1/2, t)} = \begin{cases} 1 - e^{-\left(\frac{1}{\Omega_h} + \frac{1}{\Omega_f}\right) C_4}, & t_0 \leq t \leq 1 \\ 1 - \frac{\Omega_h}{\Omega_h - \Omega_f} e^{-\frac{1}{\Omega_h} C_0 + (-\frac{1}{\Omega_f} + \frac{1}{\Omega_h}) C_4} \\ + \frac{\Omega_f}{\Omega_h - \Omega_f} e^{-\frac{1}{\Omega_f} C_0 + (-\frac{1}{\Omega_h} + \frac{1}{\Omega_f}) C_4}, & \frac{1}{2} < t \leq t_0 \end{cases} \quad (16)$$

when $\Omega_h = \Omega_f$, the outage probability of the system is as follows:

$$P_{\text{out}(\alpha=1/2, t)} = \begin{cases} 1 - e^{-\frac{2}{\Omega_h} C_4}, & t_0 \leq t \leq 1 \\ 1 - e^{-\frac{1}{\Omega_h} C_0} \\ -\frac{1}{\Omega_h} (C_0 - 2C_4) e^{-\frac{1}{\Omega_h} C_0}, & \frac{1}{2} \leq t \leq t_0 \end{cases} \quad (17)$$

C. Arbitrary time sharing and rate allocation

Finally, let time-sharing parameter and rate allocation parameter take any values. Using the similar approach as in the previous two subsections, we can obtain a generic outage probability formula as:

$$P_{\text{out}(\alpha, t)} = \begin{cases} 1 - e^{-\frac{1}{\Omega_h} C_X - \frac{1}{\Omega_f} C_Y}, & \text{if } \max\left(\frac{\alpha}{t}, \frac{1-\alpha}{1-t}\right) + \max\left(\frac{\alpha}{1-t}, \frac{1-\alpha}{t}\right) > \frac{1}{t} \\ 1 - \frac{\Omega_h}{\Omega_h - \Omega_f} e^{-\frac{1}{\Omega_h} C_0 + (-\frac{1}{\Omega_f} + \frac{1}{\Omega_h}) C_Y} \\ + \frac{\Omega_f}{\Omega_h - \Omega_f} e^{-\frac{1}{\Omega_f} C_0 + (-\frac{1}{\Omega_h} + \frac{1}{\Omega_f}) C_X}, & \text{if } \max\left(\frac{\alpha}{t}, \frac{1-\alpha}{1-t}\right) + \max\left(\frac{\alpha}{1-t}, \frac{1-\alpha}{t}\right) \leq \frac{1}{t} \end{cases} \quad (18)$$

In the case of $\Omega_h = \Omega_f$:

$$P_{\text{out}(\alpha, t)} = \begin{cases} 1 - e^{-\frac{1}{\Omega_h} (C_X + C_Y)}, & \text{if } \max\left(\frac{\alpha}{t}, \frac{1-\alpha}{1-t}\right) + \max\left(\frac{\alpha}{1-t}, \frac{1-\alpha}{t}\right) > \frac{1}{t} \\ 1 - e^{-\frac{1}{\Omega_h} C_0} - \frac{1}{\Omega_h} (C_0 - C_X) - C_Y e^{-\frac{1}{\Omega_h} C_0}, & \text{if } \max\left(\frac{\alpha}{t}, \frac{1-\alpha}{1-t}\right) + \max\left(\frac{\alpha}{1-t}, \frac{1-\alpha}{t}\right) \leq \frac{1}{t} \end{cases} \quad (19)$$

where $C_X = \max(C_{X1}, C_{X2})$, $C_Y = \max(C_{Y1}, C_{Y2})$.

Before leaving this section, we briefly discuss the conditions to reach the minimal outage probability. By observing the events (9), the constraint on R can be rewritten as:

$$R \leq \min\left(\frac{t}{\alpha} \log_2(1 + EX), \frac{1-t}{1-\alpha} \log_2(1 + EX), \frac{t}{1-\alpha} \log_2(1 + EY), \frac{1-t}{\alpha} \log_2(1 + EY), t \log_2(1 + EX + EY)\right). \quad (20)$$

As a result, under equal time-sharing condition, choosing $\alpha = 1/2$ can minimize outage probability regardless of the values of Ω_h and Ω_f . Similarly, letting $t = 1/2$ can achieve the minimal outage probability when the rate allocation is fair, for any Ω_h and Ω_f .

IV. FINITE-SNR DMT

At finite SNR, we recall the definitions of multiplexing gain r and diversity gain d in [11] as:

$$r = \frac{R}{\log_2(1 + g\gamma)}, \quad (21)$$

$$d(r, \gamma) = -\frac{\partial \ln P_{\text{out}}(r, \gamma)}{\partial \ln \gamma}. \quad (22)$$

The multiplexing gain r is the ratio of rate R to the capacity of an AWGN channel at SNR γ with array gain g chosen as the number of receive antennas at low to medium SNR. Furthermore, for a given r , diversity gain is defined as the negative slope of the log-log plot of outage probability versus SNR. In this paper, since the noise is assumed to have unit

variance, d becomes the function of both multiplexing gain r and transmission power E as follows:

$$d(r, E) = -\frac{\partial \ln P_{\text{out}}(r, E)}{\partial \ln E} = -\frac{E}{P_{\text{out}}(r, E)} \frac{\partial P_{\text{out}}(r, E)}{\partial E} \quad (23)$$

As in Section III, we derive finite-SNR DMT for different values of time sharing and rate allocation in the following three subsections.

A. Equal time sharing and arbitrary rate allocation

Substituting (21) into C_1 and C_0^* defined in Section III-A for equal time sharing, we have $C_1 = \frac{(1+E)^{2\alpha r}-1}{E}$ and $C_0^* = \frac{(1+E)^{2r}-1}{E}$. Then, substituting (12) and (13) into (23), and taking derivative, we can obtain:

$$d(r, E)_{(\alpha, t=1/2)} = \begin{cases} \frac{\left(\frac{1}{\Omega_h} + \frac{1}{\Omega_f}\right) C_{1_der}}{1 - e^{\left(\frac{1}{\Omega_h} + \frac{1}{\Omega_f}\right) C_1}}, & \alpha_0 \leq \alpha \leq 1 \\ \frac{d_1 - d_2}{P_{\text{out}}(\alpha, t=1/2)}, & \frac{1}{2} \leq \alpha \leq \alpha_0 \end{cases} \quad (24)$$

for $\Omega_h \neq \Omega_f$, and

$$d(r, E)_{(\alpha, t=1/2)} = \begin{cases} \frac{\frac{2}{\Omega_h} C_{1_der}}{1 - e^{\frac{2}{\Omega_h} C_1}}, & \alpha_0 \leq \alpha \leq 1 \\ \frac{\frac{1}{\Omega_h} (2C_{1_der} + \frac{1}{\Omega_h} (C_0^* - 2C_1) C_0^*_{-der})}{1 - e^{\frac{1}{\Omega_h} C_0^* + \frac{1}{\Omega_h} (C_0^* - 2C_1)}}, & \frac{1}{2} \leq \alpha \leq \alpha_0 \end{cases} \quad (25)$$

for $\Omega_h = \Omega_f$. Here,

$$\begin{aligned} d_1 &= \frac{\Omega_h}{\Omega_h - \Omega_f} e^{-\frac{1}{\Omega_h} C_0^* + \left(-\frac{1}{\Omega_f} + \frac{1}{\Omega_h}\right) C_1} \\ &\quad \left(-\frac{1}{\Omega_h} C_{0_der} + \left(-\frac{1}{\Omega_f} + \frac{1}{\Omega_h}\right) C_{1_der}\right), \\ d_2 &= \frac{\Omega_f}{\Omega_h - \Omega_f} e^{-\frac{1}{\Omega_f} C_0^* + \left(-\frac{1}{\Omega_h} + \frac{1}{\Omega_f}\right) C_1} \\ &\quad \left(-\frac{1}{\Omega_f} C_{0_der} + \left(-\frac{1}{\Omega_h} + \frac{1}{\Omega_f}\right) C_{1_der}\right). \end{aligned}$$

C_{1_der} and $C_{0_der}^*$ are the products of the derivatives of C_1 and C_0^* multiplied by E :

$$\begin{aligned} C_{1_der} &= \frac{2\alpha r(1+E)^{2\alpha r-1}E - (1+E)^{2\alpha r} + 1}{E}, \\ C_{0_der}^* &= \frac{2r(1+E)^{2r-1}E - (1+E)^{2r} + 1}{E}. \end{aligned}$$

When E tends to infinity, it can be shown that (24) and (25) approach:

$$d(r, E)_{(\alpha, t=1/2)} \xrightarrow{E \rightarrow \infty} \begin{cases} 1 - 2\alpha r, & 0 \leq r \leq \frac{1}{2\alpha}, \text{ if } \alpha_0 < \alpha \leq 1 \\ 1 - 2r, & 0 \leq r \leq \frac{1}{2}, \text{ if } \frac{1}{2} \leq \alpha \leq \alpha_0 \end{cases} \quad (26)$$

It is seen that the maximal multiplexing gain is smaller than one due to the additional sum-rate constraint to enforce relay

node to successfully decode both signals from two sources in MA phase.

From (24) and (25), we let $r \rightarrow 0$. Then, we can obtain the maximal diversity gain as:

$$d(r, E) \xrightarrow{r \rightarrow 0} 1 - \frac{E}{(1+E) \ln(1+E)}. \quad (27)$$

The asymptotic result in (27) agrees with the conventional point-to-point single-antenna link in [11].

B. Arbitrary time sharing and equal rate allocation

Using the same method as in the previous subsection, we can get finite-SNR DMT under fair rate allocation:

$$d(r, E)_{(\alpha=1/2, t)} = \begin{cases} \frac{d_3 - d_4}{P_{\text{out}}(\alpha=1/2, t)}, & 0 \leq t \leq \frac{1}{2} \\ \frac{d_5 - d_6}{P_{\text{out}}(\alpha=1/2, t)}, & \frac{1}{2} < t \leq t_0 \\ \frac{\left(\frac{1}{\Omega_h} + \frac{1}{\Omega_f}\right) C_{4_der}}{1 - e^{\left(\frac{1}{\Omega_h} + \frac{1}{\Omega_f}\right) C_4}}, & t_0 < t \leq 1 \end{cases} \quad (28)$$

for $\Omega_h \neq \Omega_f$, and

$$\begin{aligned} d(r, E)_{(\alpha=1/2, t)} &= \begin{cases} \frac{\frac{1}{\Omega_h} (2C_{3_der} + \frac{1}{\Omega_h} (C_0 - 2C_3) C_{0_der})}{1 - e^{\frac{1}{\Omega_h} C_0} + \frac{1}{\Omega_h} (C_0 - 2C_3)}}, & 0 \leq t \leq \frac{1}{2} \\ \frac{\frac{1}{\Omega_h} (2C_{4_der} + \frac{1}{\Omega_h} (C_0 - 2C_4) C_{0_der})}{1 - e^{\frac{1}{\Omega_h} C_0} + \frac{1}{\Omega_h} (C_0 - 2C_4)}}, & \frac{1}{2} < t \leq t_0 \\ \frac{\frac{2}{\Omega_h} C_{4_der}}{1 - e^{\frac{2}{\Omega_h} C_4}}, & t_0 < t \leq 1 \end{cases} \end{aligned} \quad (29)$$

for $\Omega_h = \Omega_f$. Here coefficients d_3 and d_5 have the same form as d_1 except that C_0^* is replaced by C_0 , C_1 is replaced by C_3 and C_4 , respectively, and C_{1_der} is replaced by C_{3_der} and C_{4_der} , correspondingly. Similarly, d_4 and d_6 have the same form as d_2 with C_0^* replaced by C_0 , C_1 replaced by C_3 and C_4 , respectively, and C_{1_der} replaced by C_{3_der} and C_{4_der} , correspondingly. Furthermore, C_{3_der} and C_{4_der} are the products of derivatives of C_3 and C_4 multiplied by E , respectively.

When E tends to infinity, we can get the DMT at asymptotically high SNR as follows:

$$d(r, E)_{(\alpha=1/2, t)} \xrightarrow{E \rightarrow \infty} \begin{cases} 1 - \frac{r}{t}, & 0 \leq r \leq t \\ 1 - \frac{r}{2(1-t)}, & \text{if } 0 \leq t \leq \frac{1}{2} \text{ or } \max\left(\frac{1}{2}, t_0\right) < t \leq \frac{2}{3} \\ & 0 \leq r \leq 2(1-t) \\ & \text{if } \frac{1}{2} < t \leq t_0 \text{ or } \max\left(\frac{2}{3}, t_0\right) < t \leq 1 \end{cases} \quad (30)$$

C. Arbitrary time sharing and rate allocation

In the most general case where the time sharing and rate allocation are arbitrary, the finite-SNR DMT formula can be

shown as

$$d(r, E)_{(\alpha, t)} = \begin{cases} \frac{\frac{1}{\Omega_h} C_{X_der} + \frac{1}{\Omega_f} C_{Y_der}}{1 - e^{-\frac{1}{\Omega_h} C_X + \frac{1}{\Omega_f} C_Y}}, & \text{if } \max\left(\frac{\alpha}{t}, \frac{1-\alpha}{1-t}\right) + \max\left(\frac{\alpha}{1-t}, \frac{1-\alpha}{t}\right) > \frac{1}{t} \\ \frac{d_7 - d_8}{P_{out}(\alpha, t)}, & \text{if } \max\left(\frac{\alpha}{t}, \frac{1-\alpha}{1-t}\right) + \max\left(\frac{\alpha}{1-t}, \frac{1-\alpha}{t}\right) \leq \frac{1}{t} \end{cases} \quad (31)$$

for $\Omega_h \neq \Omega_f$, and

$$d(r, E)_{(\alpha, t)} = \begin{cases} \frac{\frac{1}{\Omega_h} (C_{X_der} + C_{Y_der})}{1 - e^{-\frac{1}{\Omega_h} (C_X + C_Y)}}, & \text{if } \max\left(\frac{\alpha}{t}, \frac{1-\alpha}{1-t}\right) + \max\left(\frac{\alpha}{1-t}, \frac{1-\alpha}{t}\right) > \frac{1}{t} \\ \frac{\frac{1}{\Omega_h} (C_{X_der} + C_{Y_der}) + \frac{1}{\Omega_h} (C_0 - C_X - C_Y) C_{0_der}}{1 - e^{-\frac{1}{\Omega_h} C_0} + \frac{1}{\Omega_h} (C_0 - C_X - C_Y)}, & \text{if } \max\left(\frac{\alpha}{t}, \frac{1-\alpha}{1-t}\right) + \max\left(\frac{\alpha}{1-t}, \frac{1-\alpha}{t}\right) \leq \frac{1}{t} \end{cases} \quad (32)$$

for $\Omega_h = \Omega_f$, where C_{Y_der} and C_{X_der} are the products of derivatives of C_Y and C_X multiplied by E , and

$$d_7 = \frac{\Omega_h}{\Omega_h - \Omega_f} e^{-\frac{1}{\Omega_h} C_0} + \left(-\frac{1}{\Omega_f} C_X + \frac{1}{\Omega_h} C_Y\right) \left(-\frac{1}{\Omega_h} C_{0_der} - \frac{1}{\Omega_f} C_{X_der} + \frac{1}{\Omega_h} C_{Y_der}\right),$$

$$d_8 = \frac{\Omega_f}{\Omega_h - \Omega_f} e^{-\frac{1}{\Omega_f} C_0} + \left(-\frac{1}{\Omega_h} C_X + \frac{1}{\Omega_f} C_Y\right) \left(-\frac{1}{\Omega_f} C_{0_der} - \frac{1}{\Omega_h} C_{X_der} + \frac{1}{\Omega_f} C_{Y_der}\right).$$

V. NUMERICAL RESULTS

This section presents some numerical results of outage probability and finite-SNR DMT. We assume that all the three terminals are in a straight line and the relay is between S_1 and S_2 . The distance between S_1 and S_2 is fixed at 1. Let D and $1 - D$ represent the distances from S_1 to R and from S_2 to R, respectively. The path loss factor as in urban areas is assumed. Thus, Ω_h and Ω_f can be expressed as a function of D : $\Omega_h = D^{-4}$ and $\Omega_f = (1 - D)^{-4}$. For comparison, the outage probability and finite-SNR DMT results for two-phase AF protocol derived in [12] are also shown. Note that the results of AF protocol are for equal time sharing and equal rate allocation.

Fig. 3 compares the outage probabilities and finite-SNR DMTs of AF and DF under equal rate allocation and equal time sharing with $D_1 = 0.3$. It is seen that when the multiplexing gain r is small ($r \leq 0.5$), or E is small enough, the outage probability of DF is better. However, when $r > 0.5$ and E is large enough, AF outperforms. In term of finite-SNR DMT, obviously, AF is always higher than DF regardless of values of SNR. For each protocol, its finite-SNR DMT converges to the asymptotically high SNR as E increases.

Fig. 4 shows the results of varying rate allocation conditions, when $\alpha = 0.5$, both outage probability and finite-SNR DMT achieve the optimum with equal time sharing. While Fig. 5

Fig. 3. Outage probabilities and finite-SNR DMTs of AF and DF under $t = 0.5$, $\alpha = 0.5$, $D = 0.3$

presents the similar conclusion that $t = 0.5$ makes both outage probability and finite-SNR DMT reach the optimal values under fair rate allocation condition.

Finally, Fig. 6 lists the optimal relay locations to minimize outage probability at different time sharing and rate allocation. It is seen that when α is around $1/2$ or t is around $1/2$, the optimal relay location is always at $D = 0.5$. In the special case of one-way relaying, i.e., $\alpha = 0$ or 1 , the relay should be closer to the transmitting node when less time is spent on the first phase (hop) and vice versa.

VI. CONCLUSION

This paper obtains closed-form expressions for outage probability and finite-SNR DMT with any rate allocation and time sharing in the two-phase DF TWRC. Through the analysis, we can conclude that in single-antenna TWRC, DF outperforms AF when either multiplexing gain or SNR is small enough, otherwise, AF is better in term of outage probability. However, finite-SNR DMT of DF is lower than that of AF regardless of SNR due to the sum-rate constraint for relay to fully decode. Moreover, numerical results demonstrate the optimum relay location under given conditions on rate allocation and time sharing. In the future work, the performance limits of multiple-antenna TWRC with fading will be investigated.

(a) $D = 0.3, r = 0.2$

(b) $D = 0.3, 10 \log_{10} E = 10 \text{ dB}$

Fig. 4. Outage probability and finite-SNR DMT under equal time sharing and arbitrary rate allocation

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(a) $D = 0.3, r = 0.2$

(b) $D = 0.3, 10 \log_{10} E = 10 \text{ dB}$

Fig. 5. Outage probability and finite-SNR DMT under arbitrary time sharing and fair rate allocation

$\alpha \backslash D$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	0.9	0.7	0.6	0.6	0.5	0.4	0.4	0.3	0.1
0.1	0.8	0.7	0.6	0.6	0.5	0.4	0.4	0.3	0.2
0.2	0.8	0.7	0.6	0.5	0.5	0.5	0.4	0.3	0.2
0.3	0.7	0.6	0.6	0.5	0.5	0.5	0.4	0.4	0.3
0.4	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4
0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
0.6	0.4	0.4	0.5	0.5	0.5	0.5	0.5	0.6	0.6
0.7	0.3	0.4	0.4	0.5	0.5	0.5	0.6	0.6	0.7
0.8	0.2	0.3	0.4	0.5	0.5	0.5	0.6	0.7	0.8
0.9	0.2	0.3	0.4	0.4	0.5	0.6	0.6	0.7	0.8
1	0.1	0.3	0.4	0.4	0.5	0.6	0.6	0.7	0.9

Fig. 6. Optimal relay location to minimize outage probability under varying time sharing and rate allocation, $r = 0.2, 10 \log_{10} E = 10 \text{ dB}$

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