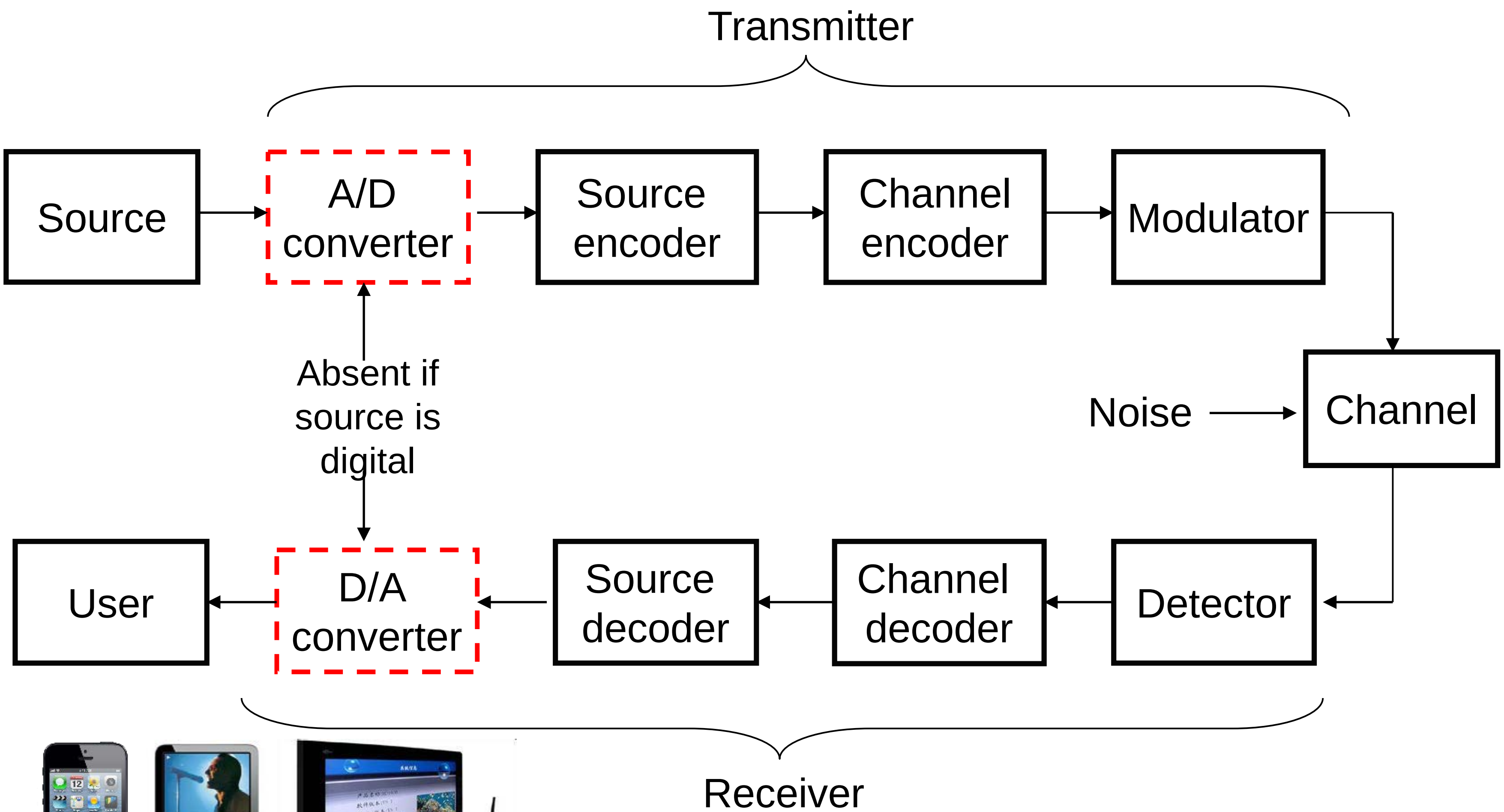
Principles of Communications

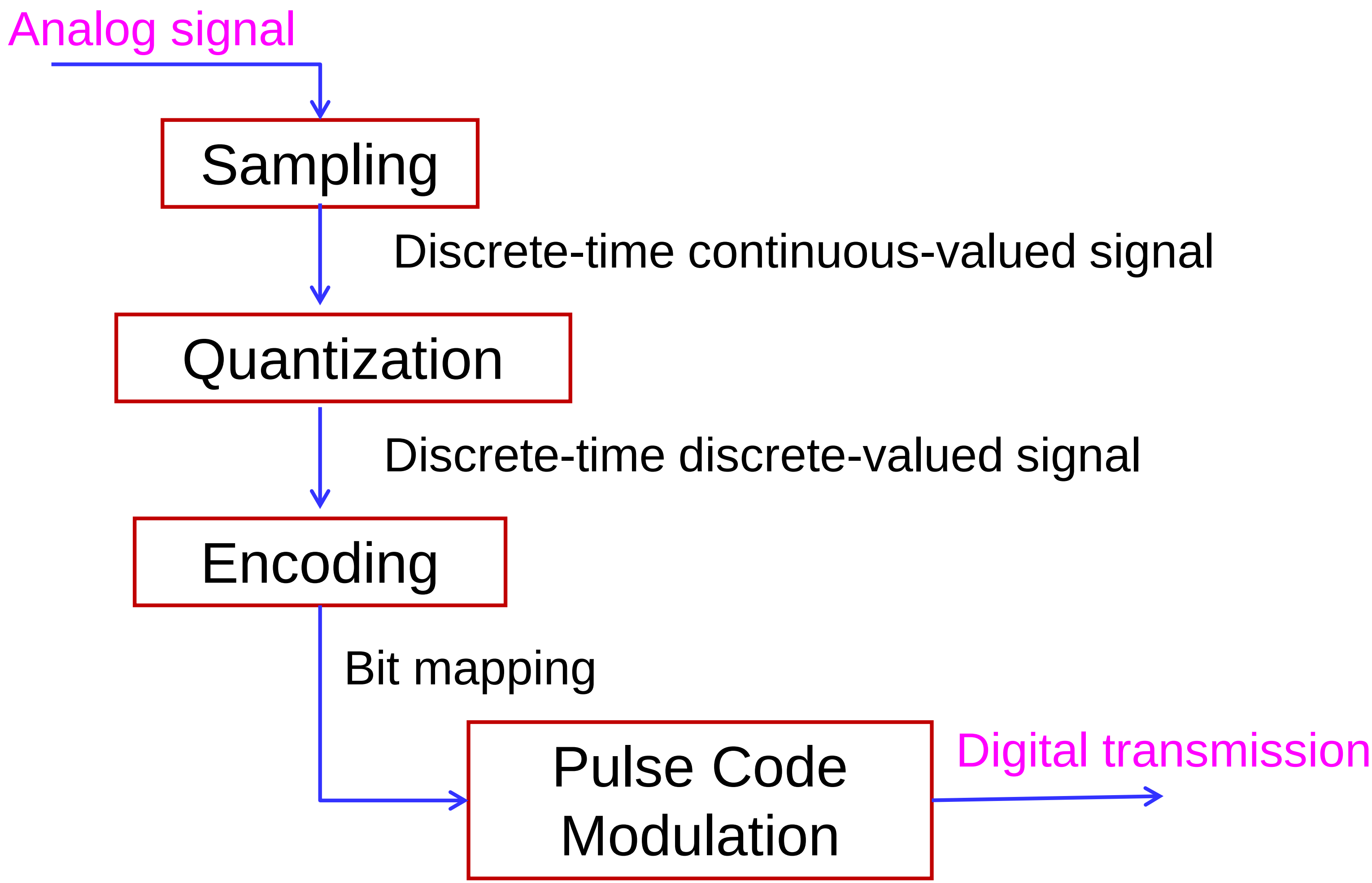
Chapter 4: Analog-to-Digital Conversion

Selected from: Chapter 7.1 – 7.4 of *Fundamentals of Communications Systems*, Pearson Prentice Hall 2005, by Proakis & Salehi

Digital Communication Systems



Topics to be Covered



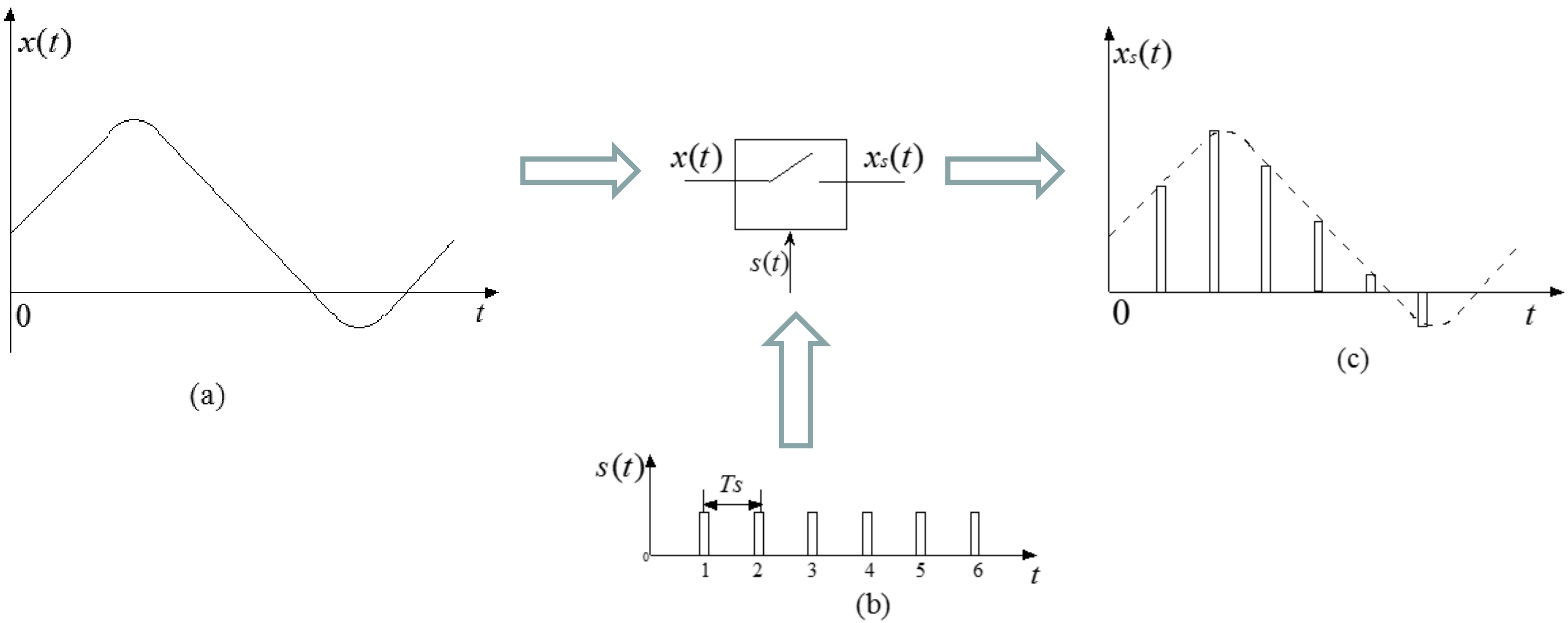
Harry Nyquist (1928)

"Certain topics in telegraph transmission theory", Trans. AIEE, vol. 47, pp. 617–644, Apr. 1928

Sampling Theorem

Continuous-time ⇨ Discrete-time

Sampling Process



$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s) = x(t)\sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

The sampling process can be regarded a modulation process with carrier given by periodic impulses. It's also called pulse modulation

Sampling Process

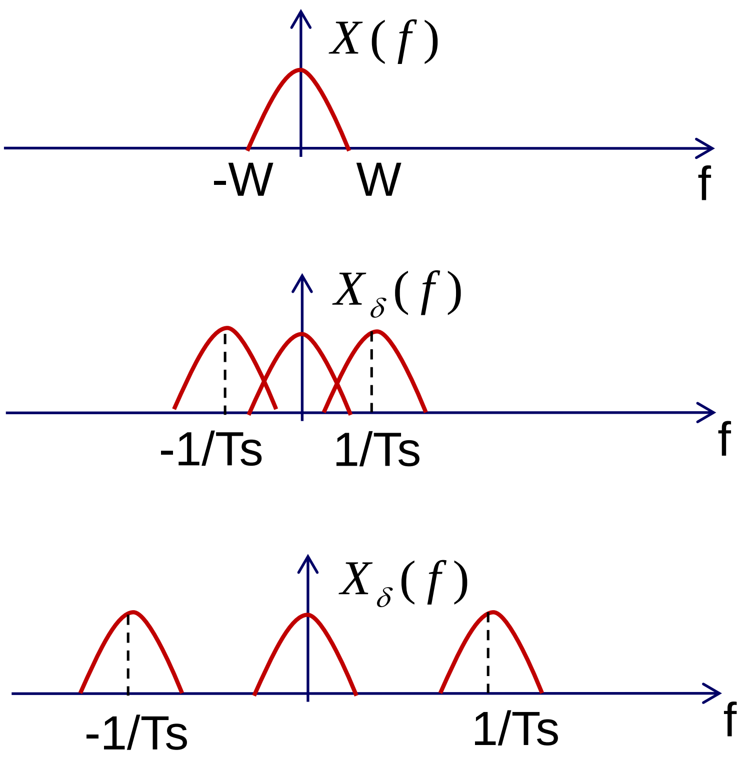
$$X_\delta(f) = X(f) * \frac{1}{T_s} \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T_s}\right) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X\left(f - \frac{n}{T_s}\right)$$

- If $f_s = \frac{1}{T_s} < 2W$ or $T_s > \frac{1}{2W}$

  aliasing error, reconstruction is not possible,

- The minimum sampling rate is known as **Nyquist sampling rate**

  $$f_s = 2W$$



Reconstruction

- LPF with frequency response

$$H(f)=\begin{cases} T_s & |f|<W \\ 0 & |f|\geq \dfrac{1}{T_s}-W \end{cases}$$

- Ideal LPF

$$H(f)=T_s\Pi\left(\frac{f}{2W'}\right) \Leftrightarrow h(t)=2W'T_s\mathrm{sinc}(2W't)$$

where $W \leq W' < \frac{1}{T_s} - W$

Reconstruction

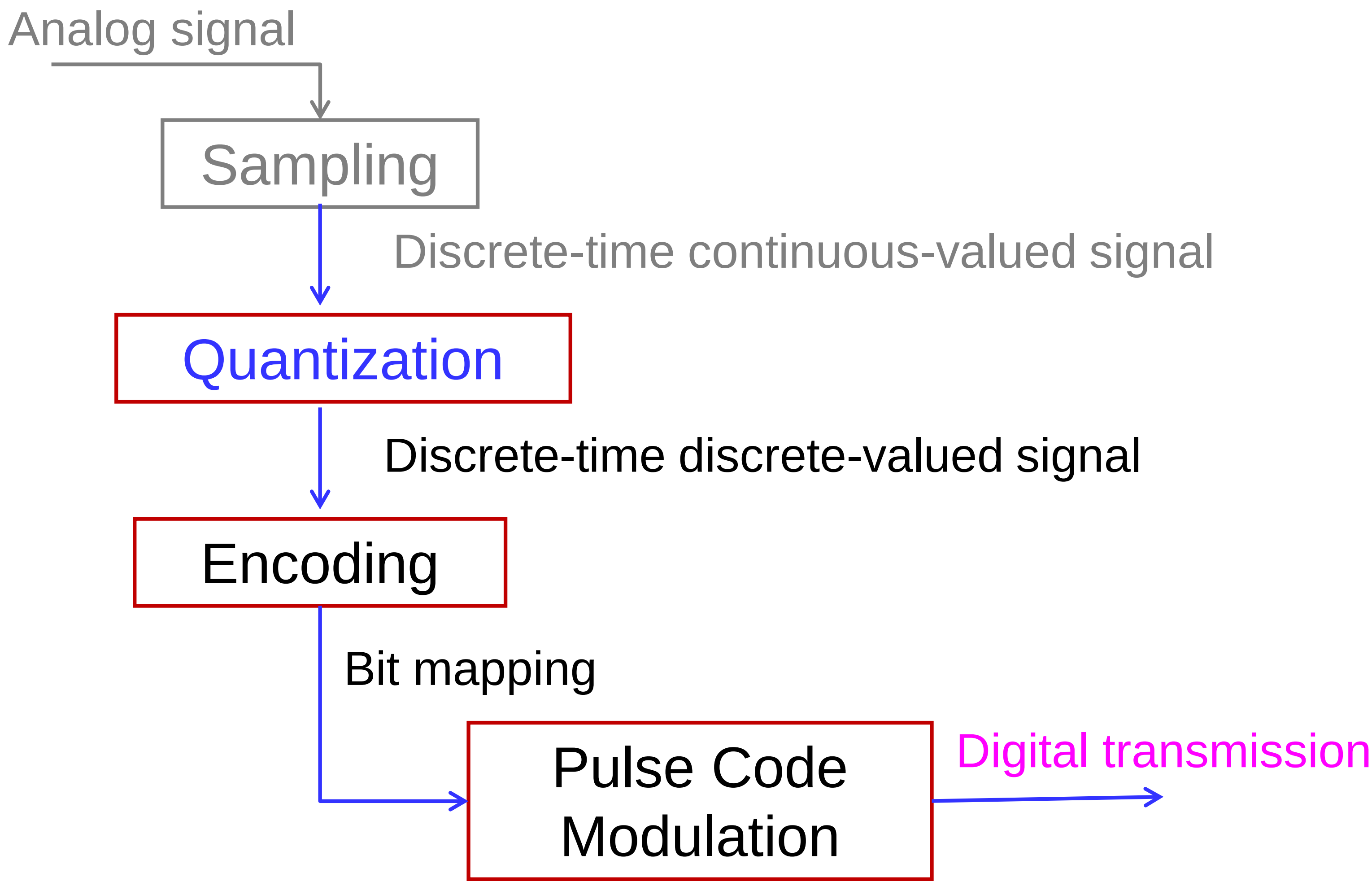
- With this choice, we have

$$x(t) = x_\delta(t) * 2W'T_s \mathrm{sinc}(2W't)$$

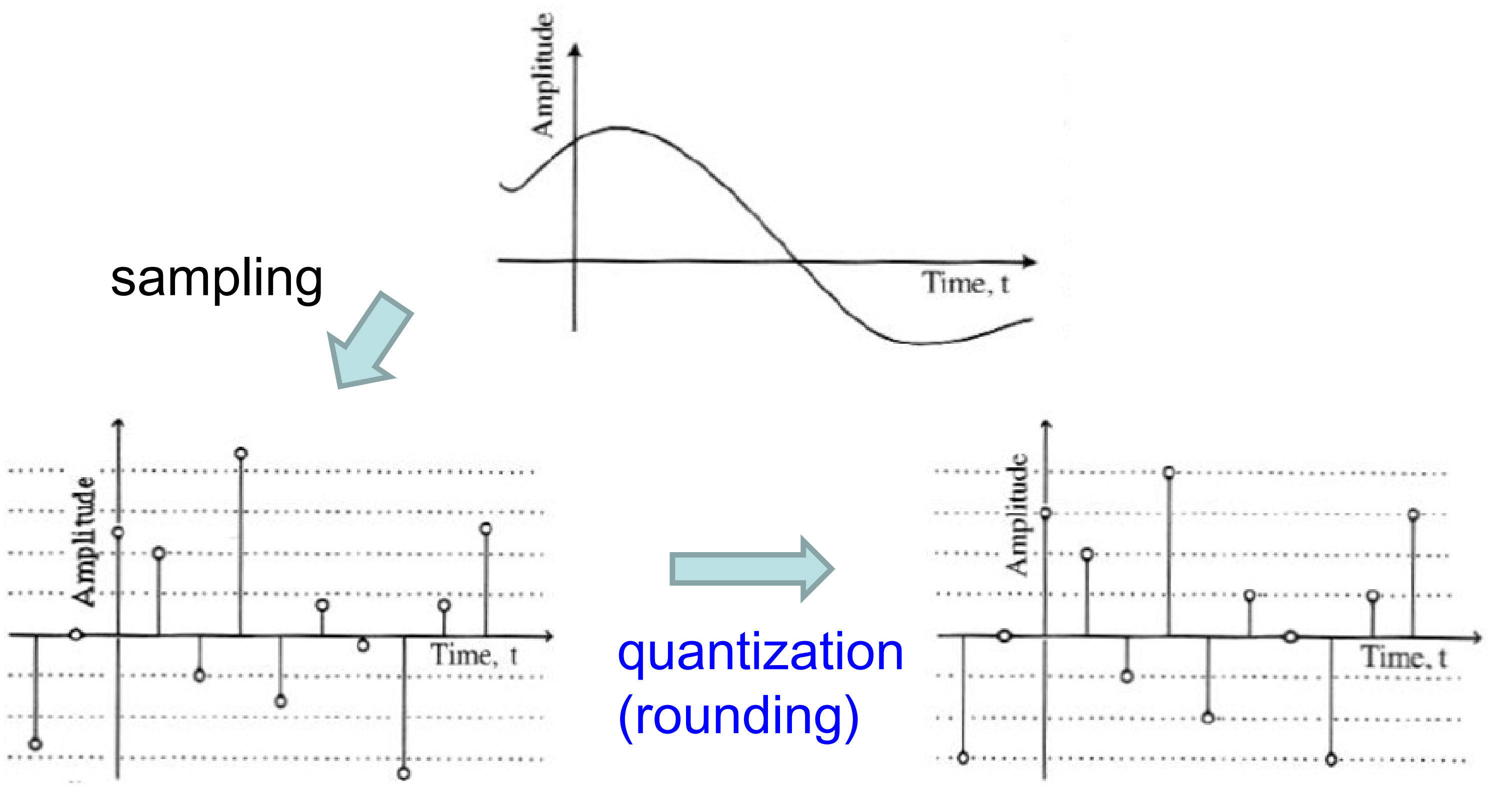
$$= \left(\sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s) \right) * 2W'T_s \mathrm{sinc}(2W't)$$

$$= \sum_{n=-\infty}^{\infty} 2W'T_s x(nT_s) \mathrm{sinc}\left[2W'(t - nT_s)\right]$$

Topics to be Covered



Quantization



Quantization Region

- Quantization regions: $\mathcal{R}_k,\ k = 1, \ldots, N$
- Quantization level for each $\mathcal{R}_k$: x_k
- Quantization:

$$Q(x) = x_k, \text{ for all } x \in \mathcal{R}_k, k = 1, \ldots, N$$

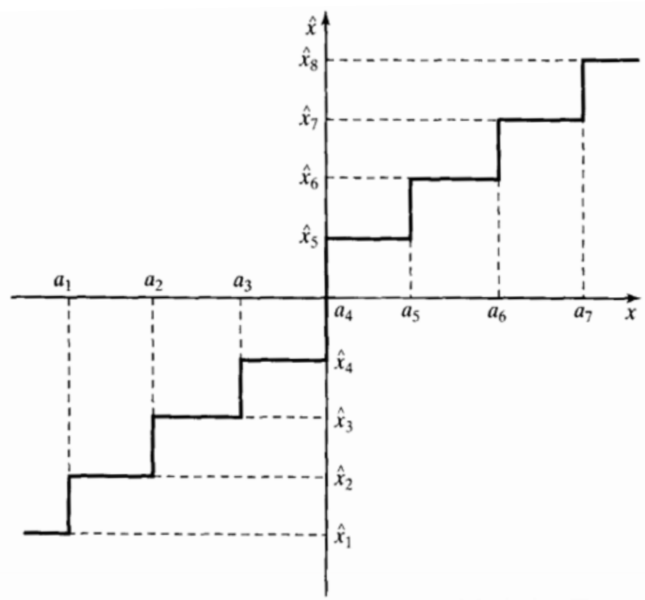


Figure 7.3 Example of an 8-level quantization scheme.

Performance Measure of Quantization

- Quantization error: $e(x, Q(x)) = x - Q(x)$
- Average (mean square error) distortion:

$$D = E\left(\left[X - Q(x)\right]^2\right)$$

- Signal-to-quantization noise ratio (SQNR):

$$SQNR = \frac{E(X^2)}{E\left(\left[X - Q(x)\right]^2\right)}$$ For random variable X

$$SQNR = \frac{\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t)dt}{\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \left(x(t) - Q(x(t))\right)^2 dt}$$ For signal x(t)

Example

- The source X(t) is a stationary Gaussian source with mean zero and power spectral density

$$S_x(f) = \begin{cases} 2 & |f| < 100Hz \\ 0 & \text{otherwise} \end{cases}$$

- It is sampled at the Nyquist rate and each sample is quantized using the 8-level quantizer with

$$a_0 = -\infty, a_1 = -60, a_2 = -40, a_3 = -20, a_4 = 0, a_5 = 20, a_6 = 40, a_7 = 60$$

$$x_1 = -70, x_2 = -50, x_3 = -30, x_4 = -10, x_5 = 10, x_6 = 30, x_7 = 50, x_8 = 70$$

- What is the resulting distortion ? Sol: 33.38
- What is the SQNR? Sol: 10.78dB
- Are the sampled signals independent to each other?

Uniform Quantizer

- Range of the input samples = [-a, a]
- Number of quantization levels = $N = 2^v$.
- Length of each quantization region $\Delta = \frac{2a}{N} = \frac{a}{2^{v-1}}$
- Quantized value are the midpoints of quantization regions.
- Quantization distortion (assume quantization error uniformly distributed on $(-\frac{\Delta}{2}, \frac{\Delta}{2})$)

$$E[e^2] = \int_{-\Delta/2}^{\Delta/2} \frac{1}{\Delta} x^2 dx = \frac{\Delta^2}{12} = \frac{a^2}{2N^2} = \frac{a^2}{3 \cdot 4^v}$$

$$SQNR = \frac{P_X}{E[e^2]} = \frac{3 \cdot 4^v P_X}{a^2} \overset{dB}{=} 10 \log_{10} \frac{P_X}{a^2} + 6v + 4.8$$

One extra bit increases the SQNR by 6 dB!

Example

- Find the SQNR for a signal uniformly distributed on $[-1, 1]$ and quantized by a uniform quantizer with 256 levels.

Nonuniform Quantizer

- By relaxing the condition that the quantization regions be of equal length, we can minimize the distortion with less constraints
- The resulting quantizer will perform better than a uniform quantizer

Optimal Quantizer

- Lloyd-Max Conditions (criteria for optimal quantizer)
  - The boundaries of the quantization regions are the midpoints of the corresponding quantized values

$$a_i = \frac{1}{2}\left(x_i + x_{i+1}\right)$$

  - The quantized values are the centroids of the quantization regions.

$$x_i = \frac{\int_{a_{i-1}}^{a_i} x f(x) dx}{\int_{a_{i-1}}^{a_i} f(x) dx}$$

[4] S. P. Lloyd, "Least-squares quantization in PCM," unpublished memorandum, Bell Laboratories, 1957 (copies available from the

[8] J. Max, "Quantizing for minimum distortion," *IRE Trans. Inform. Theory*, vol. IT-6, pp. 7-12, 1960.

TABLE 7.2 OPTIMAL NON-UNIFORM QUANTIZER FOR A GAUSSIAN SOURCE

N	$\pm a_i$	$\pm \hat{x}_i$	D	$H(\hat{X})$
1	—	0	1	0
2	0	0.7980	0.3634	1
3	0.6120	0, 1.224	0.1902	1.536
4	0, 0.9816	0.4528, 1.510	0.1175	1.911
5	0.3823, 1.244	0, 0.7646, 1.724	0.07994	2.203
6	0, 0.6589, 1.447	0.3177, 1.000, 1.894	0.05798	2.443
7	0.2803, 0.8744, 1.611	0, 0.5606, 1.188, 2.033	0.04400	2.647
8	0, 0.5006, 1.050, 1.748	0.2451, 0.7560, 1.344, 2.152	0.03454	2.825
9	0.2218, 0.6812, 1.198, 1.866	0, 0.4436, 0.9188, 1.476, 2.255	0.02785	2.983
10	0, 0.4047, 0.8339, 1.325, 1.968	0.1996, 0.6099, 1.058, 1.591, 2.345	0.02293	3.125
11	0.1837, 0.5599, 0.9656, 1.436, 2.059	0, 0.3675, 0.7524, 1.179, 1.693, 2.426	0.01922	3.253
12	0, 0.3401, 0.6943, 1.081, 1.534, 2.141	0.1684, 0.5119, 0.8768, 1.286, 1.783, 2.499	0.01634	3.372
13	0.1569, 0.4760, 0.8126, 1.184, 1.623, 2.215	0, 0.3138, 0.6383, 0.9870, 1.381, 1.865, 2.565	0.01406	3.481
14	0, 0.2935, 0.5959, 0.9181, 1.277, 1.703, 2.282	0.1457, 0.4413, 0.7505, 1.086, 1.468, 1.939, 2.625	0.01223	3.582
15	0.1369, 0.4143, 0.7030, 1.013, 1.361, 1.776, 2.344	0, 0.2739, 0.5548, 0.8512, 1.175, 1.546, 2.007, 2.681	0.01073	3.677
16	0, 0.2582, 0.5224, 0.7996, 1.099, 1.437, 1.844, 2.401	0.1284, 0.3881, 0.6568, 0.9424, 1.256, 1.618, 2.069, 2.733	0.009497	3.765
17	0.1215, 0.3670, 0.6201, 0.8875, 1.178, 1.508, 1.906, 2.454	0, 0.2430, 0.4909, 0.7493, 1.026, 1.331, 1.685, 2.127, 2.781	0.008463	3.849
18	0, 0.2306, 0.4653, 0.7091, 0.9680, 1.251, 1.573, 1.964, 2.504	0.1148, 0.3464, 0.5843, 0.8339, 1.102, 1.400, 1.746, 2.181, 2.826	0.007589	3.928
19	0.1092, 0.3294, 0.5551, 0.7908, 1.042, 1.318, 1.634, 2.018, 2.55	0, 0.2184, 0.4404, 0.6698, 0.9117, 1.173, 1.464, 1.803, 2.232, 2.869	0.006844	4.002
20	0, 0.2083, 0.4197, 0.6375, [illegible]	0.1038, 0.3128, 0.5265, [illegible]	[illegible]	[illegible]

Example

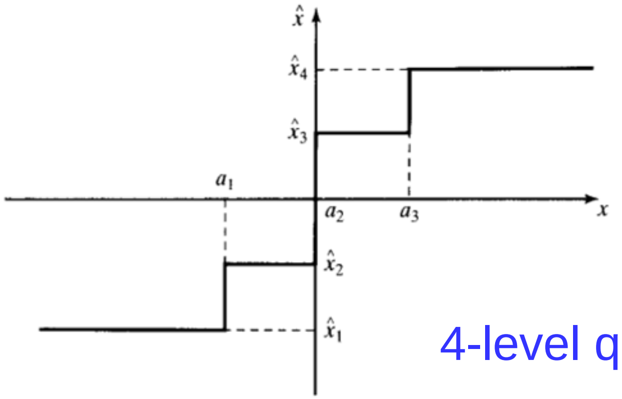
- How would the results of the example in slide#13 change if an optimal non-uniform quantizer with the same number of levels?

Sol: 14.6dB (i.e. 3.84dB better)

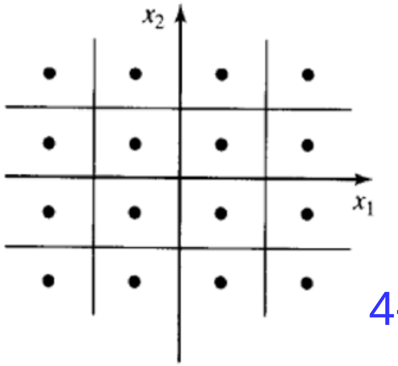
Vector Quantization

- Scalar quantization
  - Each sample is quantized individually
- Vector quantization:
  - Take blocks of source outputs of length n, and design the quantizer in the n-dim Euclidean space, rather than doing the quantization based on single samples in an one-dim space

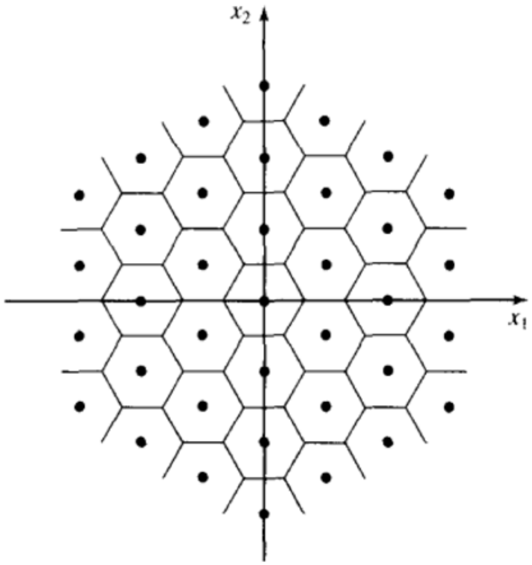
An Illustration



4-level quantizer



4-level quantizer applied to 2 samples



Vector quantization in 2-D

Optimal Vector Quantization

- Region Ri is the set of all points in the n-dim space that are closer to xi than any other xj, for all j\= i; i.e.

$$R_i = \left\{ \mathbf{x} \in R^n : \left\| \mathbf{x} - \mathbf{x}_i \right\| < \left\| \mathbf{x} - \mathbf{x}_j \right\|, \forall j \neq i \right\}$$

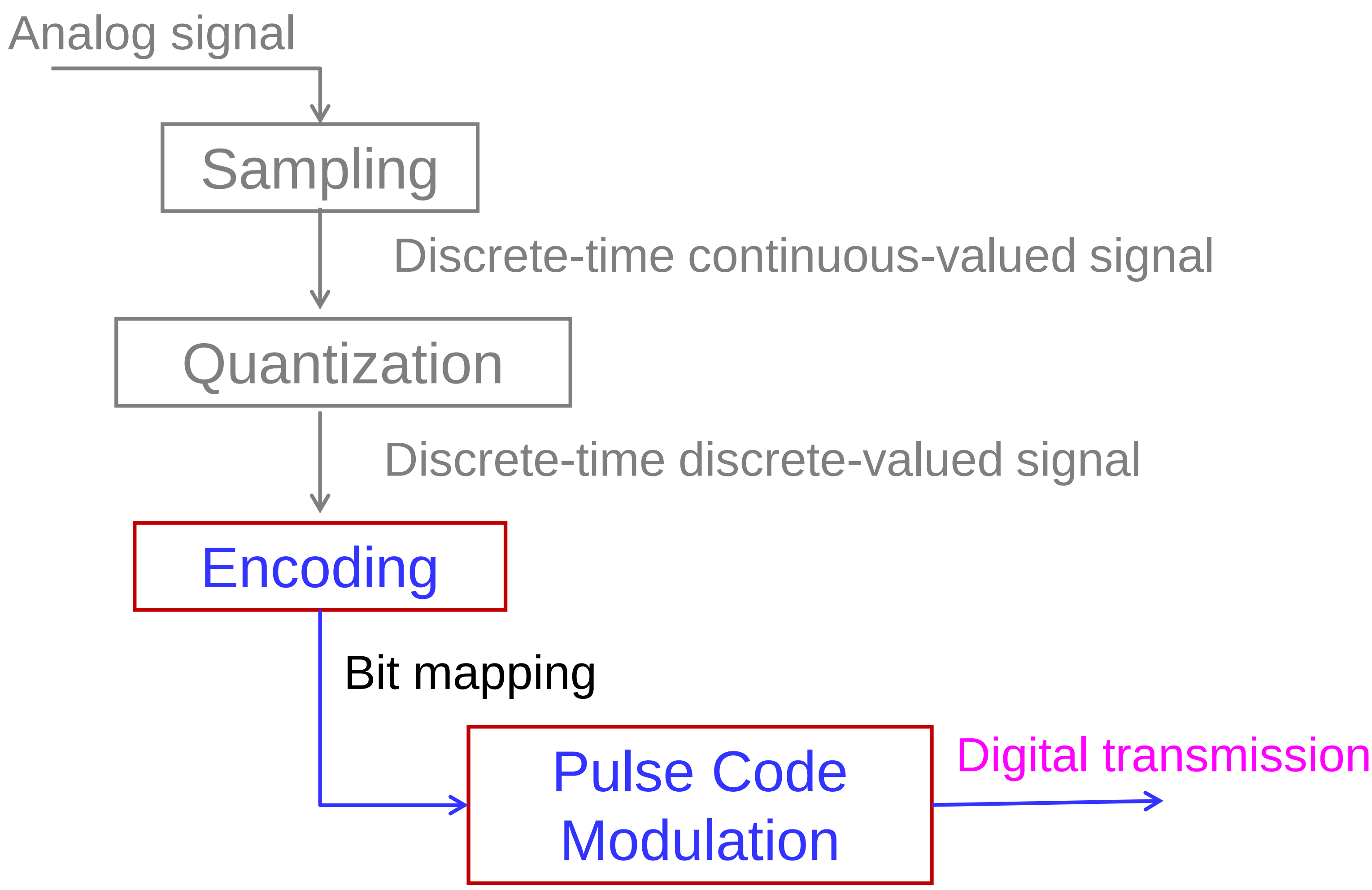
- xi is the centroid of the region Ri, i.e.

$$\mathbf{x}_i = \frac{1}{P(\mathbf{x} \in R_i)} \int_{R_i} \mathbf{x} f_{\mathbf{x}}(\mathbf{x}) d\mathbf{x}$$

Application of Quantizer

- Application of quantizer:
  - ADC in digital communication
  - Channel feedback in FDD cellular networks
- More reading:
  - Allen Gersho, "Quantization", *IEEE Communications Society Magazine*, pp. 16–28, Sept. 1977.

Topics to be Covered



Encoding

- The encoding process is to assign v bits to $N = 2^v$ quantization levels.
- Since there are v bits for each sample and fs samples/second, we have a **bit rate** of

$$R = v f_s \text{ bits/second}$$

- Natural binary coding
  - Assign the values of 0 to N-1 to different quantization levels in order of increasing level value.
- Gray coding
  - Adjacent levels differ only in one bit

Examples

- Natural binary code (NBC), folded binary code (FBC), 2-complement code (2-C), 1-complement code (1-C), and Gray code

Level no	NBC	FBC	2-C	Gray code	Amplitude level
7	111	011	011	100	3.5
6	110	010	010	101	2.5
5	101	001	001	111	1.5
4	100	000	000	110	0.5
3	011	100	100	010	-0.5
2	010	101	111	011	-1.5
1	001	110	110	001	-2.5
0	000	111	101	000	-3.5

Pulse Code Modulation (PCM)

- Most widely used method for A/D conversion of audio source
- Block diagram of a PCM system

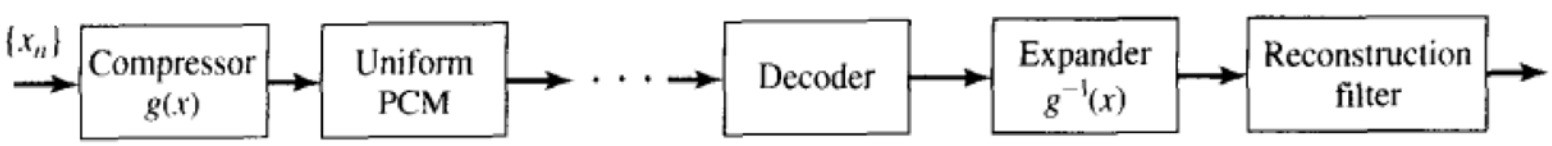
- Uniform PCM: the quantizer is a uniform quantizer
- Bit rate
  - If a signal has a bandwidth of W and is sampled at f_s and v bits are used for each sampled signal, then $R_b = f_s v \text{ bits/Hz}$
- Bandwidth requirement:
  - With binary transmission (to be discussed in later chapters)

$$BW_{req} = \frac{R_b}{2} = \frac{f_s v}{2} \geq vW \text{ Hz}$$

Non-uniform PCM

- For speech waveform, there exists a higher probability for smaller amplitudes and a lower probability for larger amplitudes.
- It makes sense to design a quantizer with more quantization regions at lower amplitudes and fewer quantization regions at large amplitudes.
- The usual method:
  - First pass the samples through a nonlinear filter that compress the large amplitudes and then perform a uniform quantization
  - Apply the inverse (expansion) of this nonlinear operation at the receiver

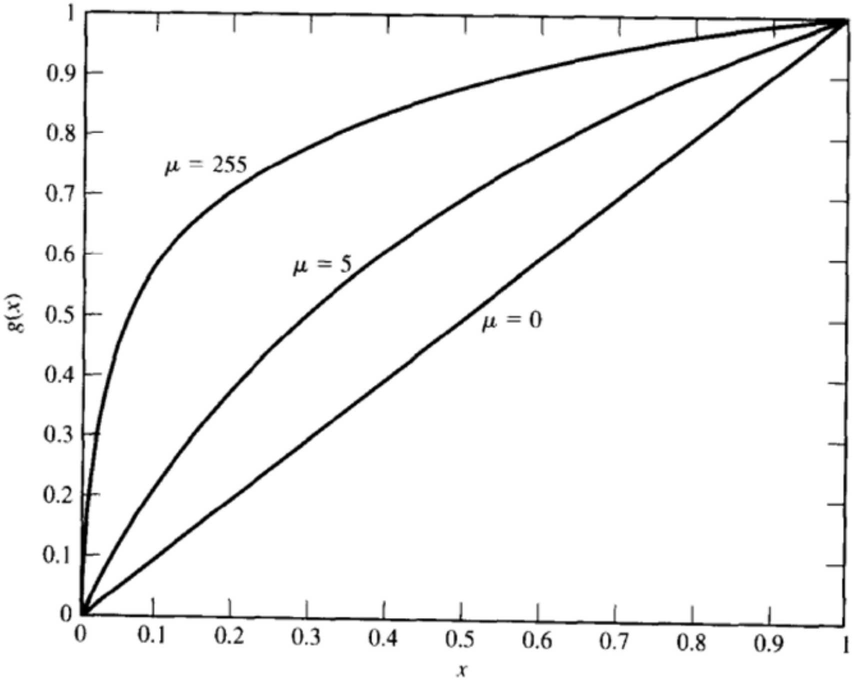
  => Companding (compressing-expanding)



Compander (speech coding)

- μ law compander

$$g(x)=\frac{\log\left(1+\mu\,|x|\right)}{\log(1+\mu)}\operatorname{sgn}(x),\quad |x|\le 1$$

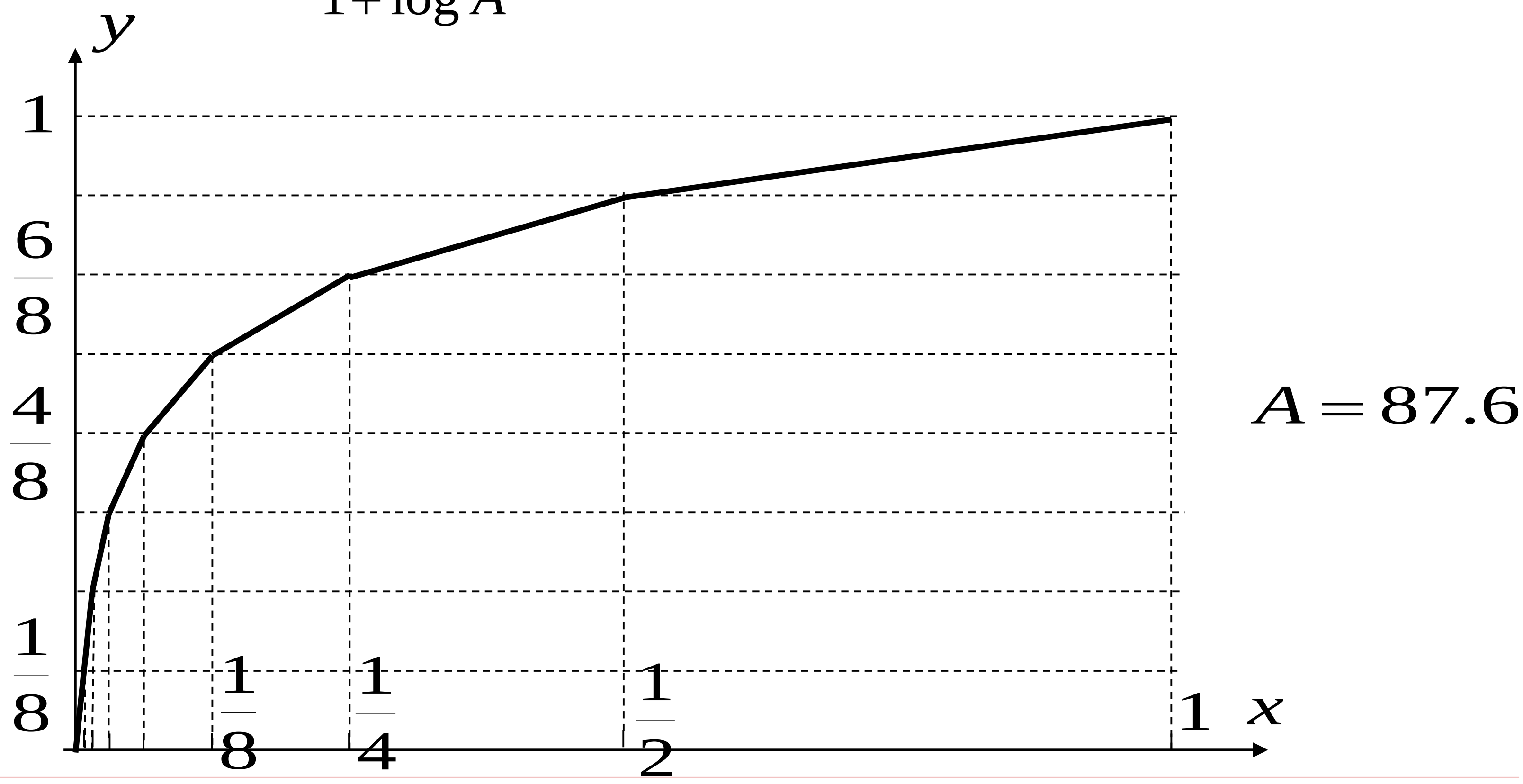


- The standard PCM system in US & Canada employs a compressor with $\mu = 255$ followed by a uniform quantizer with 8 bits/sample

Compander

- A law compander

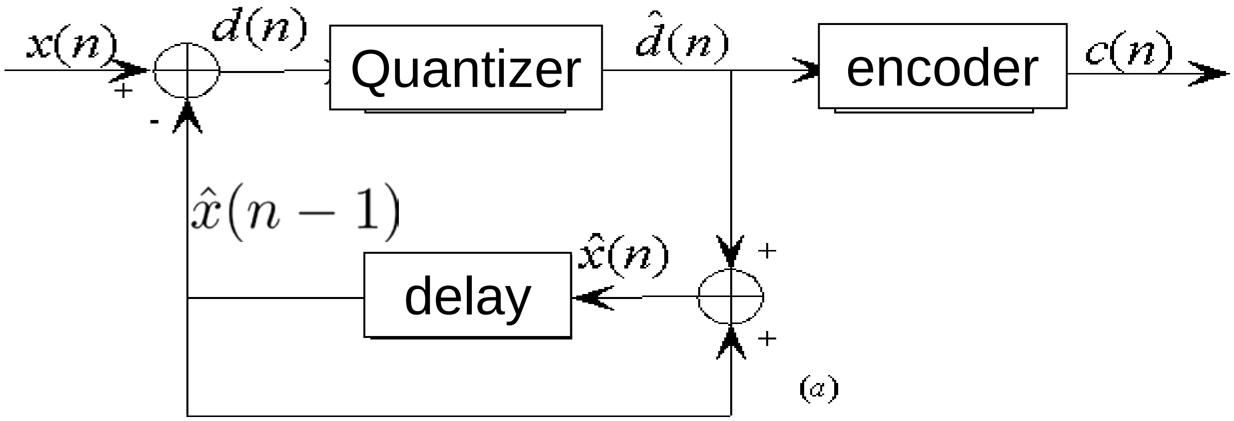
$$g(x) = \frac{1+\log A|x|}{1+\log A}\operatorname{sgn}(x), \quad |x| \le 1$$



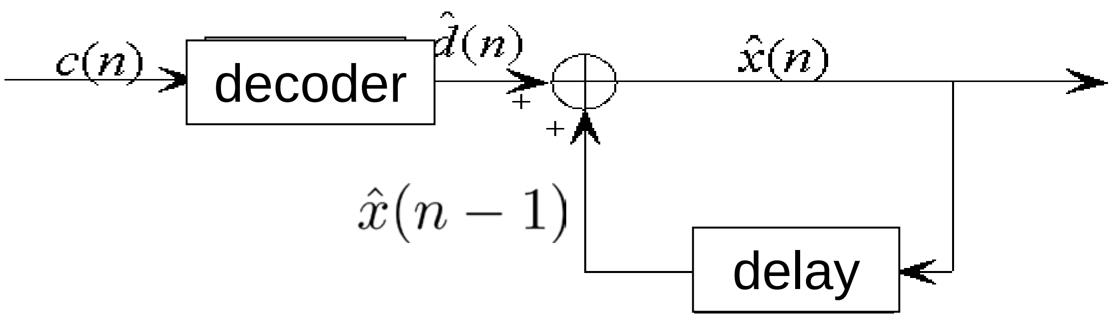
Differential PCM (DPCM)

- For a bandlimited random process, the sampled values are usually correlated random variables
- This correlation can be exploited to improve the performance
- Differential PCM: quantize the difference between two adjacent samples.
- As the difference has small variation, to achieve a certain level of performance, fewer bits are required

DPCM


$x(n)$
$d(n)$
Quantizer
$\hat{d}(n)$
encoder
$c(n)$
+
-
$\hat{x}(n-1)$
$\hat{x}(n)$
delay
+
+

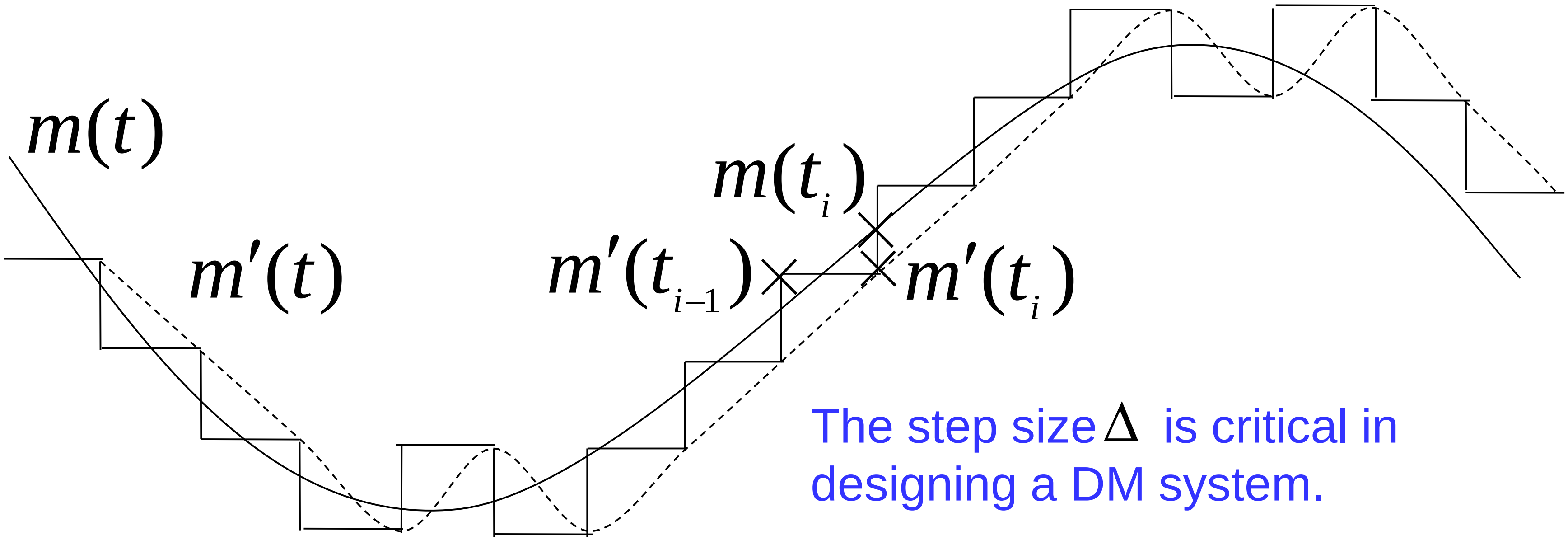
(a)


$c(n)$
decoder
$\hat{d}(n)$
$\hat{x}(n)$
+
+
$\hat{x}(n-1)$
delay

(b)

Delta Modulation (DM)

- DM is a simplified version of DPCM having a two-level quantizer with magnitude $\pm\Delta$
- In DM, only 1-bit per symbol is employed. So adjacent samples must have high correlation.



The step size Δ is critical in designing a DM system.

Step Size

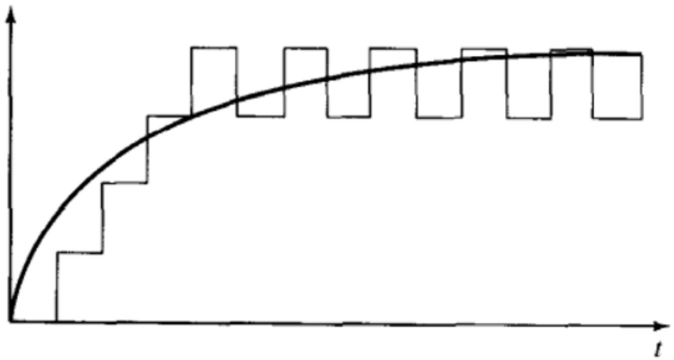


Figure 7.14 Large Δ and granular noise.

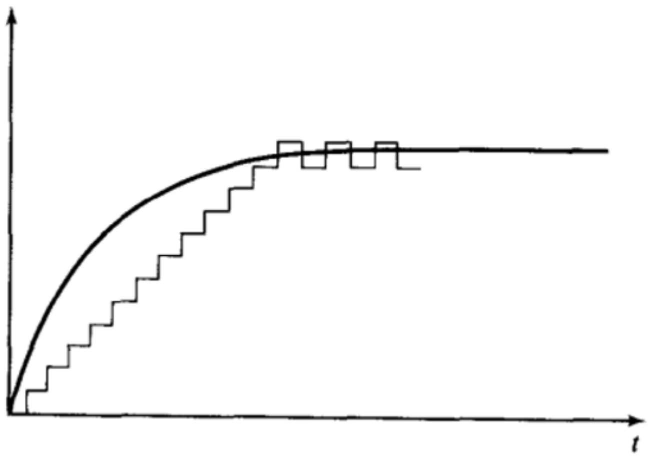


Figure 7.15 Small Δ and slope overload distortion.

Suggested Reading

- Chapter 7.1 – 7.4 of *Fundamentals of Communications Systems*, Pearson Prentice Hall 2005, by Proakis & Salehi